

Fundamentals of Diffraction and Image Formation

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I.About the authors

Course title :

Diffraction and Image Formation : Fundamentals

Module :

Interference and Diffraction

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Abstract :

After a brief reminder of 2-dimensional Fourier formalism, we will address the diffraction theory and the limitation it imposes. Then, some approximations will be made in order to simplify the calculations into simple mathematical operations. The explicit calculation will be done in detail to demonstrate one of the most remarkable and useful properties of a converging lens, namely its ability to perform 2-dimensionnal Fourier-transforms. Finally, a general study of image formation systems we will addressed by introducing the notion of optical transfer function in the case of spatially coherent and incoherent illumination.

Keywords :

Fourier Transform, Fraunhofer Diffraction, Optical transfer function, Imaging system

Prerequisites :

Fourier Analysis – Theory of linear systems

Pedagogical objectives

In the same way that it is more judicious to describe an audio amplifier response in terms of (temporal) frequencies, the objectives of the course is to teach the reader to think in terms of spatial frequencies when considering an imaging system. Moreover, this course is a fundamental prerequisite to understand the second course of this module, namely the optical filtering.

- Introduction
- Two-Dimensional Fourier Analysis
- Scalar theory of diffraction
- Fraunhofer diffraction
- Lens properties in relation with the Fourier transform
- General study of the image formation systems

Editor

Le Mans Université

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II. Lesson

Telecommunication systems - such as optical systems dedicated to image formation - are conceived to collect, treat and transport information. In the first case, information is generally encoded temporally (for example a modulated voltage), whereas in the second case it is encoded spatially (amplitude or intensity spatial profile). On a fundamental point of view, this difference is rather minimal.

A common point between the two disciplines lies in the mathematics used to describe them: system theory and Fourier analysis. The fundamental reason for this similarity is not only the word "information" but rather some properties such as *linearity* or *invariance*. Any (electronic, optic, or other) system or device which verifies those two properties can be mathematically described in a surprisingly easily fashion by using the techniques of *frequency analysis*. In the same way as it is convenient to describe amplifiers in terms of their temporal frequency responses, it is often convenient to describe image formation systems in terms of their spatial frequency responses.

It is particularly important to notice that this mathematical similarity can be exploited not only to analyze the phenomenons at stake but also to synthesize new functions. Similarly to the spectrum of a temporal function which can be intentionally manipulated using electrical filtering, the spectrum of a spatial function can be modified in diverse ways. The example of Zernike's contrast phase microscope (Noble Prize) is the best piece of evidence.

1. Two-Dimensional Fourier Analysis

Two-dimensional Fourier analysis is an extremely useful mathematical tool to study linear and nonlinear phenomenons. For a detailed study of mathematical concepts, the reader can consult, for example, references [[The Fourier integral and its application]] and/or [[The Fourier transform and its application]]. P.M. Duffieux, a French scientist, was the first to use Fourier methods to analyze optical systems [[L'intégrale de Fourier et ses applications à l'optique]].

1.1. Definitions

a) Fourier transform

The **Fourier transform** (or Fourier spectrum, or frequency spectrum) of a complex function g of 2 independent variables x and y is noted \tilde{G} or $FT(g)$, and is defined by:

$$\tilde{G}(u, v) = FT(g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(ux+vy)} dx dy$$

\tilde{G} is a function, taking complex values, of two independent variables u and v . Those variables are considered as spatial frequencies.

In a similar fashion, the inverse FT of a function $\tilde{G}(u, v)$ is noted $FT^{-1}(\tilde{G})$ and is defined by:

$$FT^{-1}(\tilde{G}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}(u, v) e^{+j2\pi(ux+vy)} du dv$$

The functions FT and FT^{-1} only differ by the sign of the exponent.

Définition

The Fourier spectrum \tilde{G} of a function g is therefore simply the ensemble of weighting factors that should be applied to each elementary function $e^{+j2\pi(u x + v y)}$ in order to restore the function g . We will come back to the physical meaning of these elementary functions later in the document.

b) Dirac delta distribution

The Dirac delta distribution (also simply called delta function) $\delta(x, y)$ is defined by its effect when used inside an integral. Mathematically, we have:

$$\iint f(x, y) \delta(x, y) dx dy = f(0, 0)$$

As a simple representation of this "function", we can consider that it takes an infinite value at the point $\{x = 0, y = 0\}$, it takes zero values everywhere else (it is defined for all real numbers), and it verifies $\int \int \delta(x, y) dx dy = 1$.

1.2. Theorems relating to Fourier transform

a) Introduction

The theorems below will be frequently used as they can greatly facilitate the search of solutions to Fourier analysis problems. Let's define :

$$FT[g(x, y)] = \tilde{G}(u, v) \text{ and } FT[h(x, y)] = \tilde{H}(u, v)$$

b) Linearity theorem

The Fourier transform of a sum of 2 functions is the sum of the 2 respective Fourier transforms:

$$FT(\alpha g + \beta h) = \alpha TF(g) + \beta TF(h)$$

c) Scaling theorem

A dilation of the spatial coordinates (x, y) results in a contraction of the frequencies u and v , and in a change in the amplitude of the whole spectrum:

$$FT[g(ax, by)] = \frac{1}{|ab|} \tilde{G}\left(\frac{u}{a}, \frac{v}{b}\right)$$

d) Transposition, conjugation and derivation

The transpose of $g(x, y)$ is $g(-x, -y)$: $FT[g(-x, -y)] = \tilde{G}(-u, -v)$.

Conjugation : $FT(g(x, y)^*) = \tilde{G}^*(-u, -v)$.

Derivation : $FT\left\{\frac{\partial^n}{\partial x^n} [g(x, y)]\right\} = (j2\pi u)^n \tilde{G}(u, v)$ pour $n \geq 1$.

e) Translation theorem

$$FT\{g(x - a, y - b)\} = \tilde{G}(u, v) e^{-j2\pi(ua + vb)}$$

A translation in the spatial domain results in a linear phase shift in the frequency domain.

f) Parseval's theorem

This theorem expresses energy conservation.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x, y)|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{G}(u, v)|^2 du dv$$

g) Convolution theorem

The convolution of 2 functions in the spatial domain is equivalent to a simple multiplication of their Fourier transforms in the frequency domain:

$$FT \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha, \beta) h(x - \alpha, y - \beta) d\alpha d\beta \right\} = TF(g * h) = \tilde{G}(u, v) \cdot \tilde{H}(u, v)$$

h) Autocorrelation theorem

This is a particular case of the previous theorem:

$$FT \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha, \beta) g^*(\alpha - x, \beta - y) d\alpha d\beta \right\} = |\tilde{G}(u, v)|^2$$

i) Reciprocity theorem

Applying successively a FT and a FT^{-1} to a function restores that function, except at the points of discontinuity:

$$FT \left\{ FT^{-1}[g(x, y)] \right\} = FT^{-1} \left\{ FT[g(x, y)] \right\} = g(x, y)$$

1.3. Separable functions

a) Introduction

A function of 2 independent variables is said to be separable if it can be expressed as a product of 2 functions, each of them depending on only one variable.

Example:

$$g(x, y) = g_x(x)g_y(y)$$

In polar coordinates:

$$g(r, \theta) = g_r(r)g_\theta(\theta)$$

These functions are easier to deal with because their two-dimensional FT simplifies in the product of 2 unidimensional FT s:

$$\begin{aligned} FT(g(x, y)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(u x + v y)} dx dy \\ &= \int_{-\infty}^{\infty} g_x(x) e^{-j2\pi(u x)} dx \int_{-\infty}^{\infty} g_y(y) e^{-j2\pi(v y)} dy \\ &= FT_x(g_x) \cdot FT_y(g_y) \end{aligned}$$

b) Functions with circular symmetry

Those functions play an important role in optics where problems often exhibit this particular symmetry. A function g is of circular symmetry if it can be written, in polar coordinates, as a function of the only variable r :

$$g(r, \theta) = g(r)$$

We recall the FT definition:

$$\tilde{G}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(u x + v y)} dx dy$$

To make use of the circular symmetry of g , we use planar polar coordinates in the planes (x, y) and (u, v) :

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \text{tg}^{-1}\left(\frac{y}{x}\right) \end{cases} ; \begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases} ; \begin{cases} \rho = \sqrt{u^2 + v^2} \\ \phi = \text{tg}^{-1}\left(\frac{v}{u}\right) \end{cases} ; \begin{cases} u = \rho \cos(\phi) \\ v = \rho \sin(\phi) \end{cases}$$

In the general case, we have:

$$FT(g) = \tilde{G}(u, v) = \tilde{G}_0(\rho, \varphi)$$

In polar coordinates, we can write:

$$u x + v y = r \rho (\cos \theta \cos \varphi + \sin \theta \sin \varphi) = r \rho \cos(\theta - \varphi)$$

$$dx dy = dS = r dr d\theta$$

consequently:

$$\tilde{G}_0(\rho, \varphi) = \int \int g_r(r) e^{-j2\pi r \rho \cos(\theta - \varphi)} r dr d\theta$$

To cover the whole plane (x, y) with x and y ranging from $-\infty$ to $+\infty$, the double integral bounds become $0 \leq r \leq \infty$ and $0 \leq \theta \leq 2\pi$. Therefore:

$$\tilde{G}_0(\rho, \varphi) = \int_0^{\infty} r g_r(r) \left\{ \int_0^{2\pi} e^{-j2\pi r \rho \cos(\theta - \varphi)} d\theta \right\} dr$$

We define the Bessel function of first kind and zero order, $J_0(a)$, where a is a non-dimensional variable, by the following integral:

$$J_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ja \cos(\theta)} d\theta$$

We can always choose the origin of angles in the plane (u, v) in order to obtain $\cos(\theta - \varphi) = \cos(\theta)$; therefore equation (I-1) becomes:

$$\tilde{G}_0(\rho, \varphi) = 2\pi \int_0^{\infty} r g_r(r) J_0(2\pi r \rho) dr = \tilde{G}_0(\rho)$$

By integrating this latter equation over r , we notice that \tilde{G}_0 depends only on ρ .

This particular form of the FT is quite common in optics. We call it **Fourier-Bessel transform** or **zero-order Hankel transform**.

A similar demonstration shows that the inverse FT of a function with a circular symmetry \tilde{G}_0 can be expressed by:

$$g_r(r) = 2\pi \int_0^{\infty} \rho \tilde{G}_0(\rho) J_0(2\pi r \rho) d\rho$$

Therefore, there is no difference between direct and inverse transforms for functions with circular symmetry. We use the notation $B\{ \}$ to represent a Fourier-Bessel transform.

Remarque

$B\{ \}$ is nothing else but a particular case of a two-dimensional FT . Therefore, any property of conventional FT s finds its analog among the properties of $B\{ \}$. In particular:

$$B\{g_r(ar)\} = \frac{1}{a^2} \tilde{G}_0\left(\frac{\rho}{a}\right)$$

1.4. Some commonly used functions and their Fourier Transforms

a) Definition

Rectangle	$\text{rect}(x) =$	$1 ; \text{ if } x < 1/2$ $0 ; \text{ elsewhere}$
Sinc (Cardinal Sine)	$\text{sinc}(x) =$	$\sin(\pi x)/(\pi x)$
Triangle	$\text{tri}(x) =$	$1- x ; \text{ if } x \leq 1$ $0 ; \text{ elsewhere}$
Circle	$\text{circ}(r) =$	$1 ; \text{ if } r \leq 1$ $0 ; \text{ elsewhere}$

b) Graphic representation

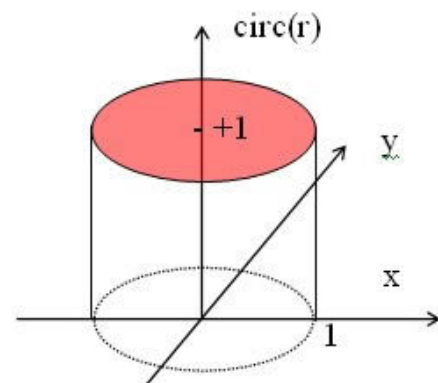
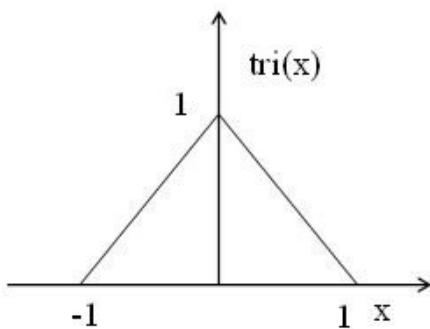
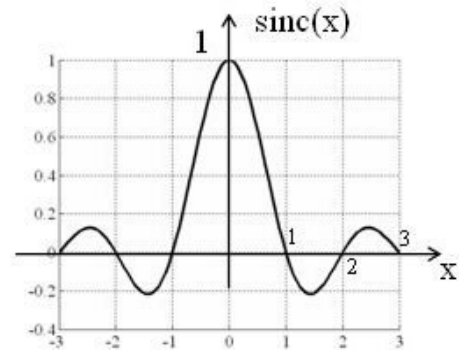
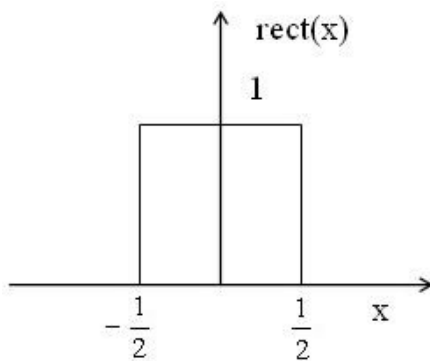


Figure I-1 - Graphical representations of some functions used

c) Fourier transforms of several separable functions in Cartesian coordinates

Function	TF
$\exp[-\pi(a^2x^2 + b^2y^2)]$	$\frac{1}{ ab } \exp\left[-\pi\left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right)\right]$
$\text{rect}(ax)\text{rect}(by)$	$\frac{1}{ ab } \text{sinc}(u/a)\text{sinc}(v/b)$
$\text{tri}(ax)\cdot\text{tri}(by)$	$\frac{1}{ ab } \text{sinc}^2(u/a)\cdot\text{sinc}^2(v/b)$
$\delta(ax, by)$	$\frac{1}{ ab }$
$\exp[j\pi(ax + by)]$	$\delta\left[u - \frac{a}{2}, v - \frac{b}{2}\right]$

d) The circle function and its Fourier transform

$\text{circ}(r) = 1$; if $r \leq 1$

= 0 ; elsewhere

The relation (I-2) for functions of circular symmetry (see paragraph "functions with circular symmetry") applies, with:

$g_r(r) = 1$ for $0 \leq r \leq 1$

By replacing (I-3) in (I-2), we obtain:

$$\tilde{G}_0(\rho) = FT[\text{circ}(r)] = 2\pi \int_0^1 r J_0(2\pi r \rho) dr$$

We note: $r' = 2\pi r \rho$; the integral bounds in relation (I-4) are equal to 0 and $2\pi\rho$.

Therefore:

$$\tilde{G}_0(\rho) = FT[\text{circ}(r)] = \frac{1}{2\pi\rho^2} \int_0^{2\pi\rho} r' J_0(r') dr'$$

Using the relation:

$$\int_0^x \alpha J_0(\alpha) d\alpha = x J_1(x)$$

where $J_1(x)$ is defined as the Bessel function of first order, we finally obtain:

$$FT[\text{circ}(r)] = J_1(2\pi\rho)/\rho$$

This function is represented on figure I-2.

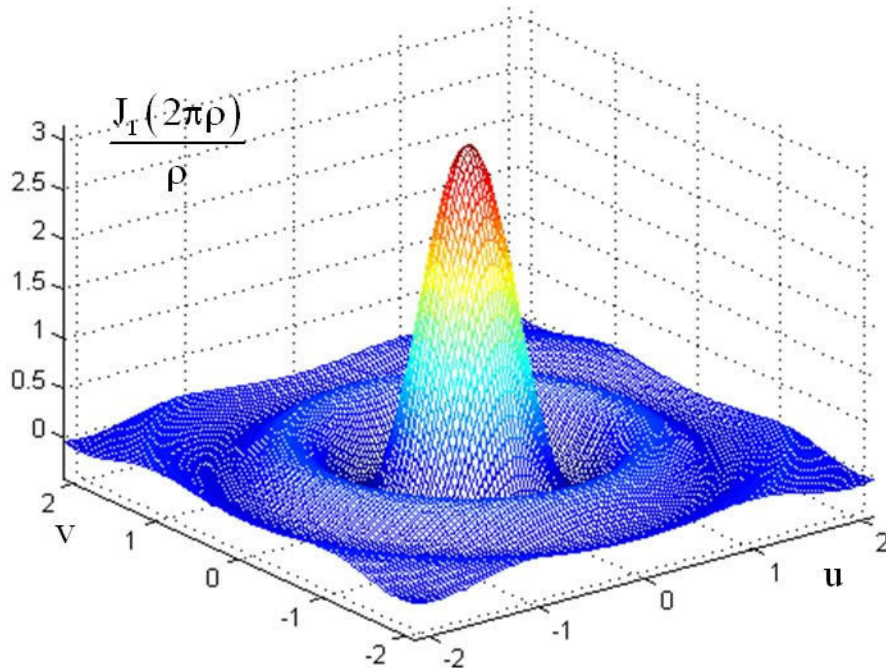


Figure I-2 - Fourier transform of the circle function

We note that the Fourier Transform of the function circle has a circular symmetry. It is composed of a central peak and a series of concentric rings of decreasing amplitudes.

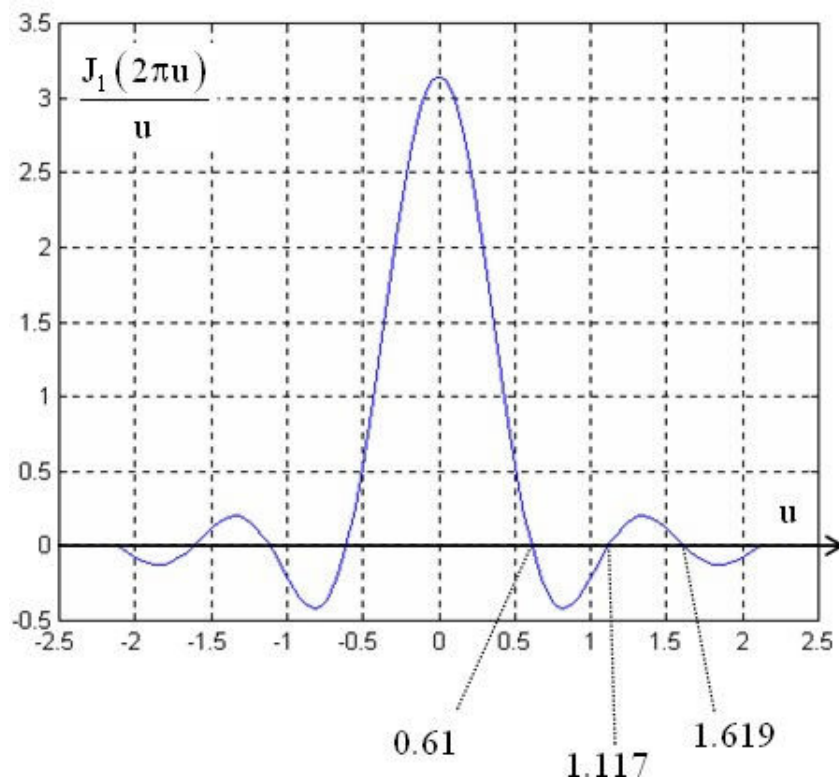


Figure I-3 - Radial profile of the function shown in Figure I-2.

Figure I-3 represents a radial profile of this function. The zeros are not regularly spaced along each ray as it is the case for the sinc (cardinal sine) function. The radius of the central peak is equal to 0.61.

2. Scalar theory of diffraction

To fully understand the physical properties of optical systems dedicated to image formation or information processing, it is essential to take into account the diffraction phenomenon and the limitations that it imposes.

2.1. Huygens-Fresnel principle

Let Σ be a diffracting surface (see figure II-1). The Huygens Fresnel principle can be expressed by :

$$U(P_0) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{e^{jkr_{01}}}{r_{01}} \cos \theta \, dS$$

where $\vec{r}_{01} = \vec{P_0P_1}$, \vec{n} is a unit vector \perp (perpendicular) to Σ , $\theta = (\vec{n}, \vec{r}_{01})$, $k = 2\pi/\lambda$, dS is an elementary surface surrounding $P_1 \in \Sigma$, and $U(P_1)$ and $U(P_0)$ are the field amplitudes in P_1 and P_0 .

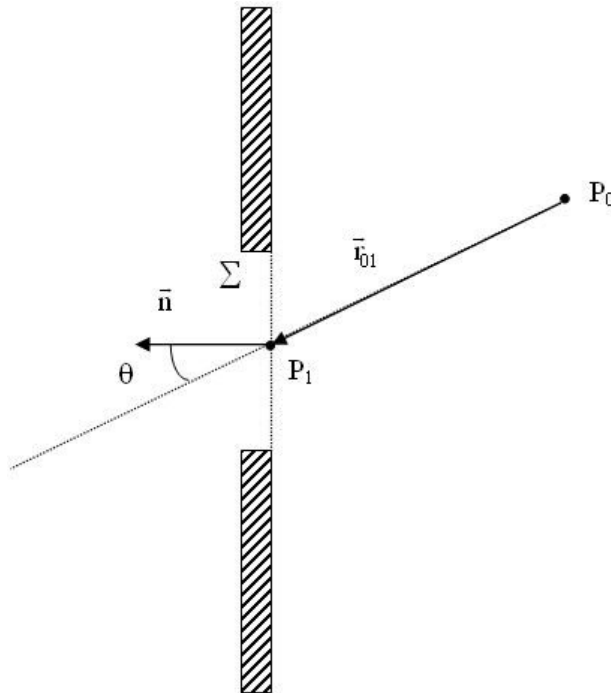


Figure II-1 - Diffractive aperture.

Physical meaning of Eq. II-1 :

The field amplitude at a point P_0 located after the aperture can be expressed as the superposition of all the diverging spherical waves $e^{jkr_{01}}/r_{01}$ emitted by the secondary sources P_1 which constitute the diffracting aperture Σ .

Any secondary source located at P_1 has the following properties:

- Its complex amplitude is proportional to the incident wave amplitude $U(P_1)$.
- It is weighted by a $\cos \theta$ term expressing its directivity with respect to the observation point, and by a term dS corresponding the elementary surface participating to the emission.

Remarque

It is important to notice that this principle is nothing else but a superposition integral expressing the system linearity: an infinity of secondary sources located on the surface Σ are interfering at P_0 .

We can write:

$$U(P_0) = \iint_{\Sigma} h(P_0, P_1) U(P_1) dS$$

where h est a weighting function defined by:

$$h(P_0, P_1) = \frac{1}{j\lambda} \frac{e^{jkr_{01}}}{r_{01}} \cos \theta$$

2.2. The angular spectrum of plane waves

Let's consider a monochromatic wave propagating along the z axis in the direction of $z > 0$ and incident on plane xOy . Let $U(x, y, 0)$ be the complex amplitude of the field at this plane. In this paragraph, we will calculate the resulting field $U(x, y, z)$ which appears at a point P_0 of coordinates (x, y, z) .

In the plane xOy , the function U has a *FT* equal to :

$$\tilde{A}_0(u, v) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} U(x, y, 0) e^{-j2\pi(u x + v y)} dx dy$$

We can consider U to be the inverse *FT* of its spectrum:

$$U(x, y, 0) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \tilde{A}_0(u, v) e^{j2\pi(u x + v y)} du dv$$

The equation of a plane wave of unit amplitude and which propagates along the unit vector \vec{u} in the direction defined by the direction cosines (α, β, γ) can be written:

$$B(x, y, z) = e^{j\vec{k} \cdot \vec{r}} \text{ avec } \vec{k} = \frac{2\pi}{\lambda} \vec{u} \text{ et } \vec{u} = (\alpha, \beta, \gamma)$$

$$B(x, y, z) = e^{j \frac{2\pi}{\lambda} (\alpha x + \beta y + \gamma z)} \text{ avec } \gamma = \sqrt{1 - \alpha^2 - \beta^2}$$

In relation (II-4), the function $\exp[j2\pi(\alpha x + \beta y)]$ can be considered as the expression, in the plane $z = 0$, of a plane wave propagating in the direction of the direction cosines:

$$\alpha = \lambda u ; \beta = \lambda v ; \gamma = \sqrt{1 - (\lambda u)^2 - (\lambda v)^2}$$

The complex amplitude of this elementary plane wave is $\tilde{A}_0(u, v) du dv$.

It is for this reason that the function $\tilde{A}_0(u, v)$ previously defined in relation (II-3) can be written:

$$\tilde{A}_0\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} U(x, y, 0) e^{-\frac{j2\pi}{\lambda} (\alpha x + \beta y)} dx dy$$

It is called angular spectrum of the perturbation $U(x, y, 0)$.

2.3. Helmholtz equation

A light perturbation at point P and at instant t is represented by the scalar function $u(P, t)$. For a monochromatic wave, we can explicitly express it as:

$$u(P, t) = A(P) \cos(2\pi\nu t + \phi(P))$$

$A(P)$ and $\varphi(P)$ are respectively the amplitude and phase of the wave at point P ; ν is the temporal frequency. Using the complex notation:

$$u(P, t) = \operatorname{Re}\{U(P)e^{-j2\pi\nu t}\} \text{ where } U(P) = A(P)e^{-j\phi(P)}$$

$U(P)$ is the complex amplitude.

If the real perturbation $u(P, t)$ represents an optical wave, it must satisfy the following scalar wave equation at any space point where there is no source:

$$\Delta \cdot u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

where Δ is the Laplace operator: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and c is the speed of light.

Since the dependence with t is known *a priori*, the knowledge of the complex function $U(P)$ is sufficient to describe the perturbation. By replacing (II-7) in (II-8), we deduce that the complex amplitude must obey the following equation:

$$(\Delta + k^2)U = 0$$

The relation (II-9) is known as the **Helmholtz equation**. In the following, we will assume that the complex amplitude of any monochromatic optical wave propagating in free space must obey that relation

2.4. Propagation of the angular spectrum

Let us consider the angular spectrum of a wave U located at a distance z along the propagation axis (see figure II-2) :

$$\tilde{A}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x, y, z) e^{-\frac{j2\pi}{\lambda}(\alpha x + \beta y)} dx dy$$

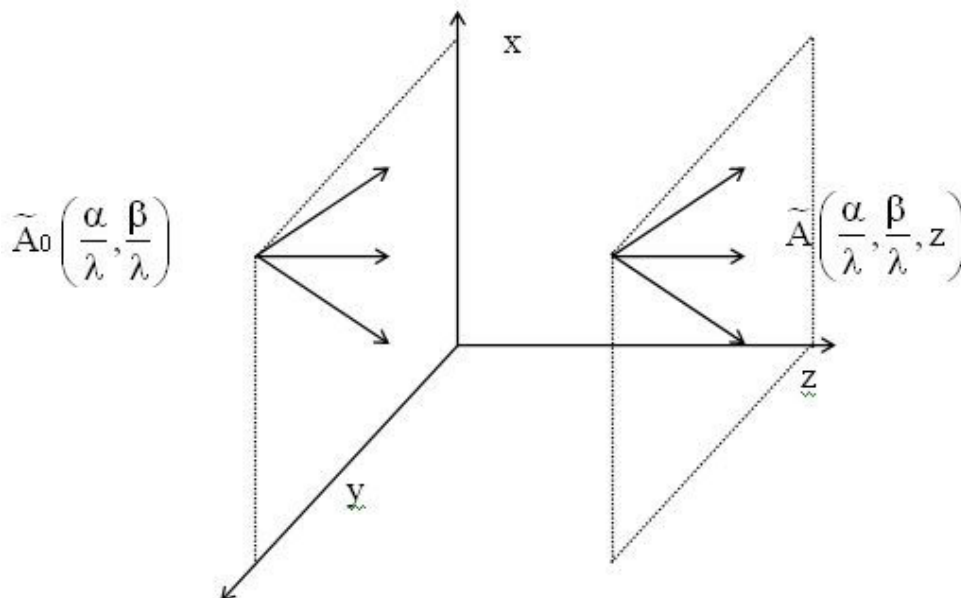


Figure II-2- Propagation of the angular spectrum along z .

To characterize the effects of wave propagation on the perturbation angular spectrum, we need to determine the relation between $\tilde{A}_0\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right)$ and $\tilde{A}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, z\right)$. We know that $U(x, y, z)$ can be written under the form of a FT^{-1} :

$$U(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, z\right) e^{+\frac{j2\pi}{\lambda}(\alpha x + \beta y)} d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right)$$

In addition, U must follow the Helmholtz equation at any point where there is no source. By replacing (II-10) in (II-9) and after calculation, we find that $\tilde{A}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, z\right)$ must satisfy the following differential equation [[Aspects of diffraction theory and their application to the ionosphere]] :

$$\frac{d^2 \tilde{A}}{dz^2} + \left(\frac{2\pi}{\lambda}\right)^2 (1 - \alpha^2 - \beta^2) \tilde{A} = 0$$

An elementary solution to this equation can be written as:

$$\tilde{A}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, z\right) = \tilde{A}_0\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) e^{j\frac{2\pi}{\lambda} \sqrt{1 - \alpha^2 - \beta^2} \cdot z}$$

Remarque

- This result shows that when the direction cosines satisfy the inequality $\alpha^2 + \beta^2 < 1$, the effect of propagation on a distance z translates into a simple phase shift of the various angular spectrum components.
- In the (rarer) case where the direction cosines verify $\alpha^2 + \beta^2 > 1$ (for example in the presence of a diopter), the square root is imaginary and the previous relation can be written:

$$\tilde{A}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, z\right) = \tilde{A}_0\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) e^{-\mu \cdot z} \text{ avec } \mu = \frac{2\pi}{\lambda} \sqrt{\alpha^2 + \beta^2 - 1}$$

Since $\mu > 0$, the spectral components are attenuated by the propagation phenomenon. Those spectral components are called "evanescent waves".

- The borderline case where $\alpha^2 + \beta^2 = 1$ corresponds to waves propagating perpendicularly to Oz . Consequently, they transport no energy along the z axis.

2.5. The propagation phenomenon considered as a linear spatial filter

The expression (II-11) can be rewritten as a function of the spatial frequencies u and v :

$$\frac{\tilde{A}(u, v, z)}{\tilde{A}_0(u, v)} = \tilde{H}(u, v) = e^{j\frac{2\pi}{\lambda} z \sqrt{1 - (\lambda u)^2 - (\lambda v)^2}}$$

The function \tilde{H} is called "**transfer function**" and characterizes the system in the frequency domain; more particularly, here, it characterizes the propagation phenomenon.

If the distance z is larger than a few wavelengths, we can neglect the evanescent waves (i.e. the case where the square root is imaginary). The transfer function then simplifies in:

$$\tilde{H}(u, v) = \begin{cases} e^{j\frac{2\pi}{\lambda} z \sqrt{1 - (\lambda u)^2 - (\lambda v)^2}}, & \text{if } u^2 + v^2 < \frac{1}{\lambda^2} \\ 0, & \text{elsewhere} \end{cases}$$

Fondamental

The propagation phenomenon can be considered as a low-pass filter, whose transmittance is equal to zero outside of a circular region of radius $\rho = \frac{1}{\lambda}$ in the frequency plane. Inside this circular domain, the transfer function modulus is equal to 1, but a phase shift dependent on (u, v) , λ and z is introduced.

2.6. Effects of a diffracting aperture on the angular spectrum of a perturbation

We place an aperture Σ in the plane xOy (at $z = 0$). We note $U_i(x, y, 0)$ the incident field just before Σ and $U_t(x, y, 0)$ the transmitted field just after Σ (see figure II-3).

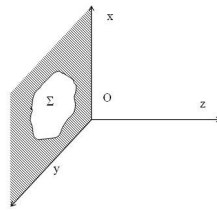


Figure II-3 - Diffracting aperture in the xOy plane

We define the transmittance $t(x, y)$ of the aperture Σ as the ratio between the complex amplitudes taken immediately after and before Σ .

$$t(x, y) = \frac{U_t(x, y, 0)}{U_i(x, y, 0)}$$

In the case of figure II-3, Σ being an aperture:

$$t(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Sigma \\ 0, & \text{elsewhere} \end{cases}$$

a) General case, valid for any incident wave

Considering the definition of t : $U_t(x, y, 0) = U_i(x, y, 0)t(x, y)$, the convolution theorem implies:

$$\tilde{A}_t\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \tilde{A}_i\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) * \tilde{T}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right)$$

with

$$\tilde{T}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t(x, y) e^{-j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)} dx dy$$

b) Particular case of a planar incident wave

In this case, the result is simpler:

$$U_i(x, y, 0) = 1 \Rightarrow \tilde{A}_i\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \delta\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right)$$

By applying (II-13):

$$\tilde{A}_t\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \tilde{T}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right)$$

Therefore, we directly obtain the angular spectrum of the transmitted wave by calculating the FT of the aperture transmittance.

Fondamental

Introducing a diffracting aperture which spatially limits the incident wave results in broadening the perturbation angular spectrum. Indeed, the smaller is the aperture and the wider are both its FT and the angular spectrum of the transmitted wave. A similar effect occurs in the temporal domain, which is characterized by the relation: $\Delta\nu\Delta t \approx 1$. The shorter the pulse duration and the wider the frequency spectrum.

3. Fraunhofer diffraction

In the previous chapter, we presented the results of diffraction scalar theory under their most general form. Here, we will make approximations that will allow simplifying the calculations into basic mathematical operations.

3.1. Approximations to the Huygens-Fresnel principle

Since $U(x_1, y_1) = 0$ outside of Σ (see figure III-1), we can rewrite the integral defined in Eq. II-2A by integrating from $-\infty$ to $+\infty$:

$$U(x_0, y_0) = \int_{-\infty}^{\infty} \int h(x_0, y_0, x_1, y_1) U(x_1, y_1) dx_1 dy_1$$

General approximations:

- distance $z \gg$ the largest linear dimension of Σ .
- distance $z \gg$ the largest linear dimension of the region of observation.

With those two hypothesis, we obtain:

$$\begin{cases} \cos(\vec{n}, \vec{r}_{01}) \approx 1 \\ h(x_0, y_0, x_1, y_1) = \frac{1}{j\lambda} \frac{e^{jkr_{01}}}{z} \end{cases}$$

The term r_{01} in the exponent cannot be replaced by z as it is done in the denominator because the error which would result from this approximation would be multiplied by $k \approx 2 \times 3.14 / (0.5 \times 10^{-6}) = 1.2 \times 10^7$ and would lead to errors on the phase far larger than 2π radians.

3.2. Fresnel approximations

$$r_{01} = z \sqrt{1 + \frac{(x_0 - x_1)^2}{z^2} + \frac{(y_0 - y_1)^2}{z^2}}$$

The Fresnel approximations consist in assuming that the first 2 terms of the square root Taylor expansion are sufficient to correctly represent the phase, provided that z is large enough :

$$r_{01} \approx \left\{ 1 + \frac{1}{2} \left(\frac{x_0 - x_1}{z} \right)^2 + \frac{1}{2} \left(\frac{y_0 - y_1}{z} \right)^2 \right\}$$

The weighting factor can then be written as :

$$h(x_0, y_0, x_1, y_1) = \frac{\exp(jkz)}{j\lambda z} \exp\left\{ j \frac{k}{2z} [(x_0 - x_1)^2 + (y_0 - y_1)^2] \right\}$$

which means that for a distance z large enough, the maximum phase change due to the first neglected term has to be $\ll 1 \text{ rad}$:

$$kz \left(\frac{\epsilon_{max}}{8} \right) \ll 1$$

This condition will be realized if :

$$z^3 \gg \frac{\pi}{4\lambda} [(x_0 - x_1)^2 + (y_0 - y_1)^2]_{Max}^2$$

The superposition integral (III-1) then becomes :

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} \int_{-\infty}^{\infty} \int U(x_1, y_1) e^{j\frac{k}{2z} [(x_0 - x_1)^2 + (y_0 - y_1)^2]} dx_1 dy_1$$

We can develop the quadratic terms in the exponent :

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z} [x_0^2 + y_0^2]} \int_{-\infty}^{\infty} \int U(x_1, y_1) e^{j\frac{k}{2z} [x_1^2 + y_1^2]} e^{-j\frac{2\pi}{\lambda z} [x_0 x_1 + y_0 y_1]} dx_1 dy_1$$

Therefore, except for the phase and amplitude multiplying factors which are independent of x_1 and y_1 , we can calculate the diffracting wave amplitude by calculating the *FT* :

$$FT \left\{ U(x_1, y_1) \exp \left\{ j\frac{k}{2} z [x_1^2 + y_1^2] \right\} \right\}$$

This *FT* has to be evaluated at frequencies :

$$u = \frac{x_0}{\lambda z} ; v = \frac{y_0}{\lambda z}$$

3.3. Fraunhofer approximations

Fraunhofer made an approximation on the quadratic phase function :

$$e^{j\frac{k}{2z} [x_1^2 + y_1^2]} \approx 1 \text{ if } z \gg \frac{k(x_1^2 + y_1^2)_{Max}}{2}$$

This condition is even more restrictive than the Fresnel condition :

For example, if $(x_1^2 + y_1^2)_{Max} = 1 \text{ mm}^2 = 10^{-6} \text{ m}^2$ and $\lambda = 0.5 \mu\text{m}$, then the Fraunhofer approximation corresponds to $z \gg 6 \text{ m}$. With the same parameters, the Fresnel conditions correspond to $z \gg 1 \text{ cm}$. For this reason, Fraunhofer diffraction is also called far-field diffraction.

Using those conditions, the expression (III-2) can be written :

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z} (x_0^2 + y_0^2)} \int_{-\infty}^{\infty} \int U(x_1, y_1) e^{-j\frac{2\pi}{\lambda z} [x_0 x_1 + y_0 y_1]} dx_1 dy_1$$

Except for some multiplying factors preceding the integral, this expression is simply the FT of the field distribution at the aperture evaluated for the frequencies $u = x_0/\lambda z$; $v = y_0/\lambda z$.

Remarque

Fraunhofer diffracting patterns can be observed at distances smaller than previously calculated if a converging lens is properly placed between the aperture and the observation plane. This is precisely what we will examine in the following section, entitled: "lens properties in relation with the Fourier transform".

4. Lens properties in relation with the Fourier transform

One of the most remarkable and most useful properties of a converging lens is its natural and extremely simple ability to perform a two-dimensional Fourier transform. This convenient property of the optical domain has no equivalence in the electronics domain, where an analogue Fourier transform operation requires cumbersome and expensive frequency analyzers.

4.1. The thin lens considered as a phase transformer

Définition

A lens is said to be thin if a ray entering through one side at point (x, y) emerges on the opposite side at the same point (x, y) (negligible ray translation).

Therefore, a thin lens simply delays the incident wave phase with a quantity proportional to its thickness at any point (x, y) (see figure IV-1).

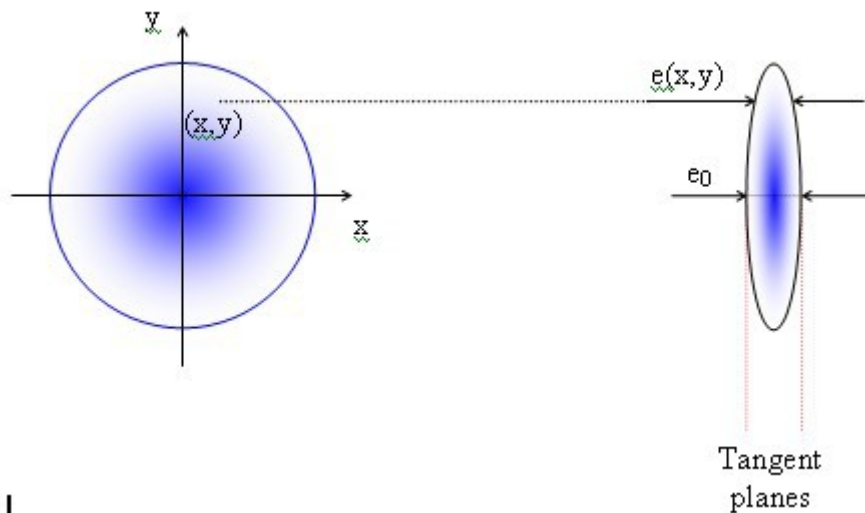


Figure IV-1 Thin lens: front and side view

Let e_0 be the maximum thickness of the lens, $\varphi_L = kne(x, y)$ the phase delay introduced by the lens and $\varphi_A = ke_0 - ke(x, y)$ the phase delay introduced by the remaining part of the free space that lies between the planes tangent to the entrance and exit sides of the lens.

The total phase delay accumulated by the wave can therefore be written:

$$\varphi(x, y) = \varphi_L + \varphi_A = k(n-1)e(x, y) + ke_0$$

Let $U_L(x, y)$ and $U'_L(x, y)$ be the complex fields located immediately in front and below the lens.

The effect introduced by the lens can therefore be described by a phase transform $U'_L(x, y) = U_L(x, y)t_L(x, y)$, where :

$$t_L(x, y) = \exp(jke_0) \cdot \exp[jk(n-1) \cdot e(x, y)]$$

A relatively simple mathematic calculation shows that the thickness function can be written (within the paraxial approximation) [[Introduction to Fourier Optics]]:

$$e(x, y) = e_0 - \frac{(x^2 + y^2)}{2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

where R_1 and R_2 are the radius of curvature of the entrance and exit sides of the lens. By replacing Eq. (IV-2) in (IV-1) we obtain:

$$t_L(x, y) = \exp(jkne_0) \cdot \exp \left[-jk(n-1) \cdot \frac{(x^2 + y^2)}{2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right]$$

If we neglect the constant phase term and we group the characteristic dimensions of the lens (n, R_1, R_2) into only one number f , called focal distance, such that:

$$\frac{1}{f} \equiv (n - 1) \cdot \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

then the phase transform can be written:

$$t_L(x, y) = e^{-\frac{jk}{2f}(x^2+y^2)}$$

This relation neglects the finite size of the lens. We will take it into account later.

Sign convention: $R > 0$ for a center of curvature to the right of the surface, and $R < 0$ in the opposite case. This sign convention, adopted for R and consequently for f , allows applying t_L to all types of converging (figure IV-2) and diverging (figure IV-3) lenses:

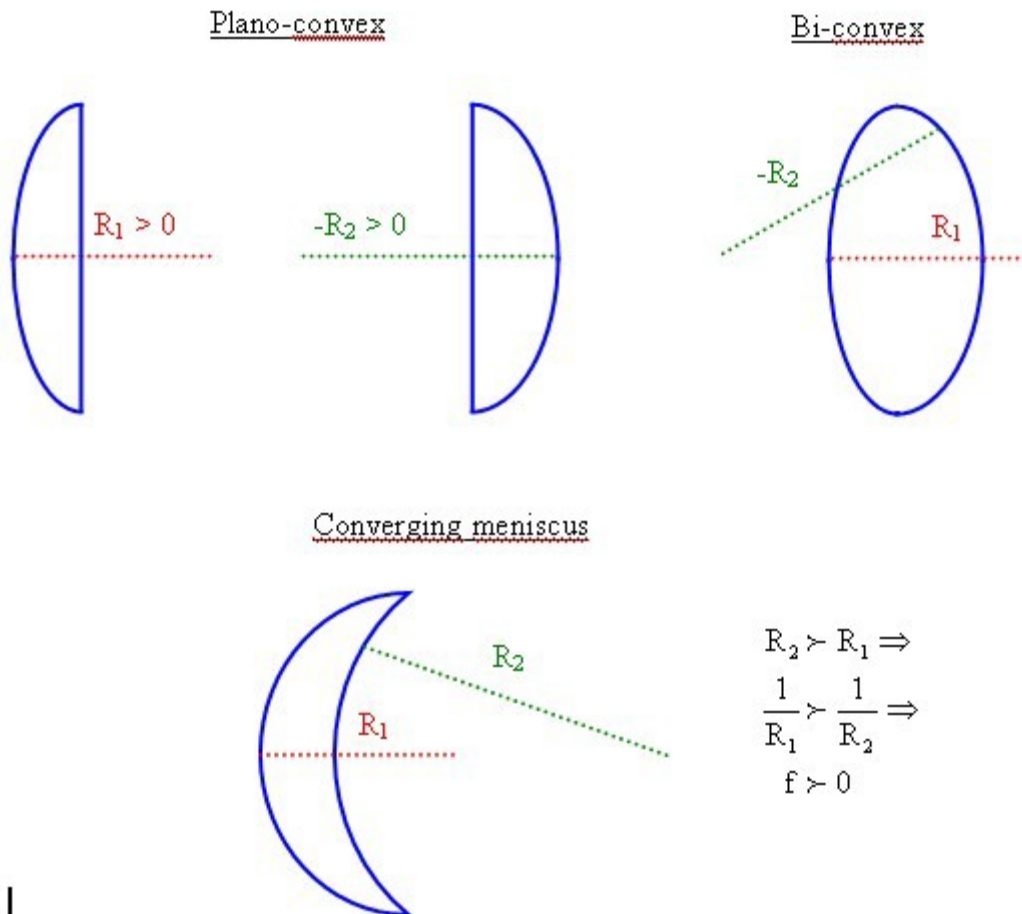


Figure IV-2- Various types of converging lenses

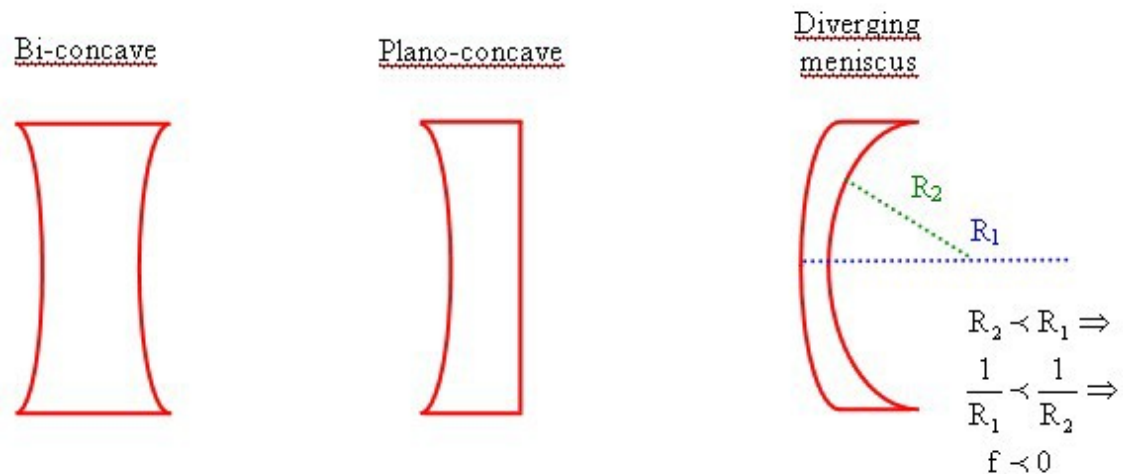


Figure IV-3- Various types of diverging lenses

4.2. Properties of lenses in relation to the Fourier transform

a) Introduction

In what follows, we will assume a monochromatic illumination and we will be interested in the amplitude distribution of light in the image focal plane of the lens. Three cases will be examined depending on the location of the object by respect to the lens.

b) Case where the object is located right near the lens

Let us consider an plane object of amplitude transmittance $t_o(x, y)$ located right before a converging lens (see figure IV-4).

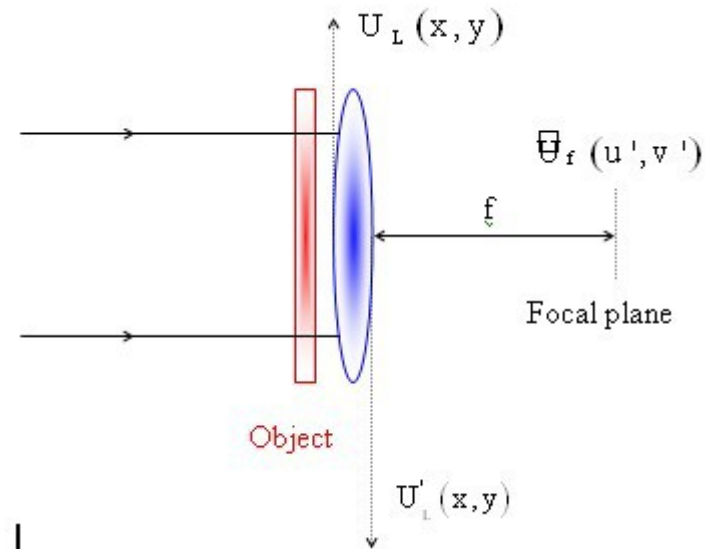


Figure IV-4- Object placed against the lens

We assume that the object is uniformly illuminated with a plane wave of amplitude A at normal incidence. In the plane tangent to the lens side we can write:

$$U_L(x, y) = A \cdot t_o(x, y)$$

The field distribution right after the back of the lens thus becomes:

$$U'_L(x, y) = U_L(x, y) \cdot e^{-\frac{jk}{2f}(x^2+y^2)} \cdot P(x, y)$$

where, to take into account the finite size of the lens, we define the pupil function:

$$P(x, y) = \begin{cases} 1, & \text{inside the lens aperture} \\ 0, & \text{elsewhere} \end{cases}$$

In order to determine the field distribution at the lens focal plane, we apply the Fresnel diffraction formula with $z = f$ (Eq. III-2) :

$$\tilde{U}_f(u', v') = \frac{e^{jkf}}{j\lambda f} e^{j\frac{k}{2f}[u'^2+v'^2]} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U'_L(x, y) e^{j\frac{k}{2f}[x^2+y^2]} \cdot e^{-j\frac{2\pi}{\lambda f}[xu'+yv']} dx dy$$

By letting aside the constant phase factor $\exp(jkf)$ and by replacing $U'_L(x, y)$ with its value, we notice that the quadratic phase terms inside the integral are compensating one other:

$$\tilde{U}_f(u', v') = \frac{e^{j\frac{k}{2f}[u'^2+v'^2]}}{j\lambda f} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_L(x, y) P(x, y) e^{-j\frac{2\pi}{\lambda f}[xu'+yv']} dx dy$$

If the object spatial dimensions are smaller than the lens aperture (see figure IV-5), the factor $P(x, y)$ disappears from the equation (since the field is equal to zero where the object is opaque, and therefore $U_L(x, y)P(x, y) = U_L(x, y)$:

$$U_f(u', v') = \frac{e^{j\frac{k}{2f}[u'^2+v'^2]}}{j\lambda f} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_L(x, y) e^{-j\frac{2\pi}{\lambda f}[xu'+yv']} dx dy$$

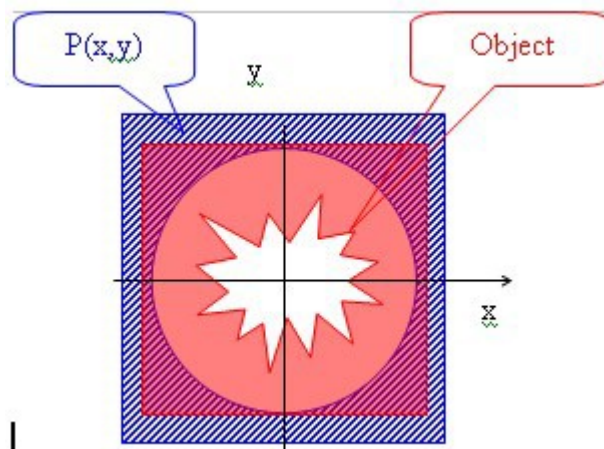


Figure IV-5- Object placed against the lens

Remarque

- We note that the complex amplitude at the focal plane is the Fraunhofer diffraction figure of the incident field in front of the lens; therefore, using a lens allows to observing Fraunhofer diffraction figure at observation distances far smaller than the ones defined in chapter 3 (here, the observation distance is equal to f).
- Apart from some multiplying factors in front of the integral, the previous expression

(IV-3) is therefore a simple Fourier transform :

$$FT[U_L(x, y)] \left(u = \frac{u'}{\lambda f}, v = \frac{v'}{\lambda f} \right)$$

- The quadratic phase term in front of the FT expresses what we call a “**phase curvature**”.
- Generally, we are interested in I_f , the intensity distribution called “**power angular spectrum**” or “**spectral energy**” of the object:

$$I_f(u', v') = \left(\frac{A}{\lambda f} \right)^2 \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_o(x, y) e^{-j \frac{2\pi}{\lambda f} [xu' + yv']} dx dy \right|^2$$

c) Object located before the lens

Let us consider an object characterized by its amplitude transmittance $t_o(x, y)$ and located at a distance d in front of the lens (see figure IV-6).

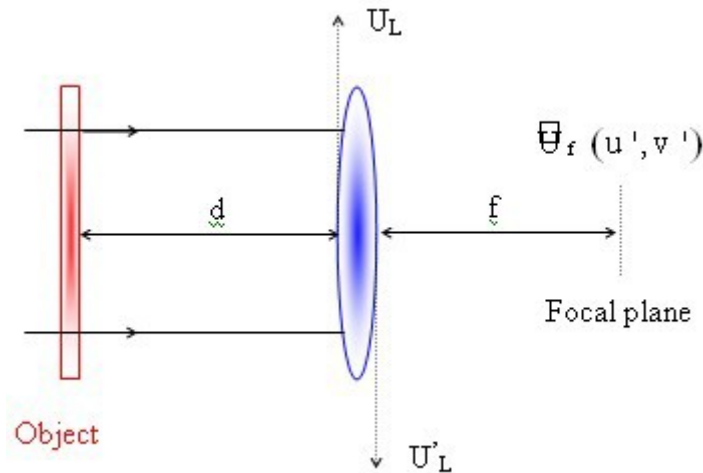


Figure IV-6- Object placed at a distance d in front of the lens

The previous setup is a particular case of this one (for $d = 0$). We assume that the object is uniformly illuminated with a plane wave of amplitude A at normal incidence.

We define : $\tilde{F}_o(u, v) = TF\{A \cdot t_o(x, y)\}$ and $\tilde{F}_L(u, v) = TF\{U_L(x, y)\}$. We know that the transfer function of the propagation phenomenon allows calculating \tilde{F}_L starting from \tilde{F}_o (Eq. II-12 in the section : Scalar theory of diffraction) :

$$\tilde{F}_L(u, v, d) = \tilde{F}_o(u, v) \cdot e^{j \frac{2\pi}{\lambda} d \sqrt{1 - (\lambda u)^2 - (\lambda v)^2}}$$

The lens will be considered in the paraxial approximation, therefore the rays are only slightly inclined with respect to the optical axis: $\alpha^2 = (\lambda u)^2 \ll 1$ and $\beta^2 = (\lambda v)^2 \ll 1$.

We make a Taylor expansion of the square root in the previous expressions and we obtain:

$$\tilde{F}_L(u, v, d) = \tilde{F}_o(u, v) \cdot e^{j \frac{2\pi}{\lambda} d} \cdot e^{-j\pi \lambda d (u^2 + v^2)}$$

$e^{j \frac{2\pi}{\lambda} d}$ is a constant phase term that can be omitted in what follows. In addition, we already characterized the propagation of $U_L(x, y)$ between a plane located right near the lens and the focal plane. It is described in the previous paragraph (Eq. IV-3):

$$\tilde{U}_f(u', v') = \frac{e^{j \frac{k}{2f} [u'^2 + v'^2]}}{j\lambda f} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_L(x, y) e^{-j \frac{2\pi}{\lambda f} [xu' + yv']} dx dy = \frac{e^{j \frac{k}{2f} [u'^2 + v'^2]}}{j\lambda f} \tilde{F}_L\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right)$$

By applying the propagation transfer function (Eq. IV-4), we obtain:

$$\tilde{F}_L\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right) = \tilde{F}_o\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right) \cdot e^{-j\pi\lambda d \left[\left(\frac{u'}{\lambda f}\right)^2 + \left(\frac{v'}{\lambda f}\right)^2 \right]}$$

In the exponent term, we factorize the term $1/(\lambda f)^2$, simplify by λ , and introduce $k = 2\pi/\lambda$:

$$\tilde{F}_L\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right) = \tilde{F}_o\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right) \cdot e^{\frac{-jkd}{2f^2}(u'^2+v'^2)}$$

Therefore the expression (IV-5) becomes:

$$\tilde{U}_f(u', v') = \frac{e^{j\frac{k}{2f}[u'^2+v'^2]}}{j\lambda f} \cdot \tilde{F}_o\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right) \cdot e^{\frac{-jkd}{2f^2}(u'^2+v'^2)}$$

By regrouping the exponential functions we find:

$$\tilde{U}_f(u', v') = \frac{1}{j\lambda f} \cdot \tilde{F}_o\left(\frac{u'}{\lambda f}, \frac{v'}{\lambda f}\right) \cdot e^{j\frac{k}{2f}(u'^2+v'^2)\left(1-\frac{d}{f}\right)}$$

or again:

$$\tilde{U}_f(u', v') = \frac{e^{j\frac{k}{2f}(u'^2+v'^2)\left(1-\frac{d}{f}\right)}}{j\lambda f} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_o(x, y) e^{-j\frac{2\pi}{\lambda f}[xu'+yv']} dx dy$$

Remarque

We note that a phase factor precedes the *FT* of the object. This phase factor vanishes for $d = f$ (if the object is placed in the front focal plane of the lens). In that case, the phase quadratic curvature disappears, leading to an exact Fourier transform.

d) Object located after the lens

The lens is illuminated with a monochromatic plane wave of amplitude A and at normal incidence (same illumination conditions as in the previous paragraphs). The object is therefore illuminated with a spherical wave converging towards the focal plane of the lens in the image space (see figure IV-7).

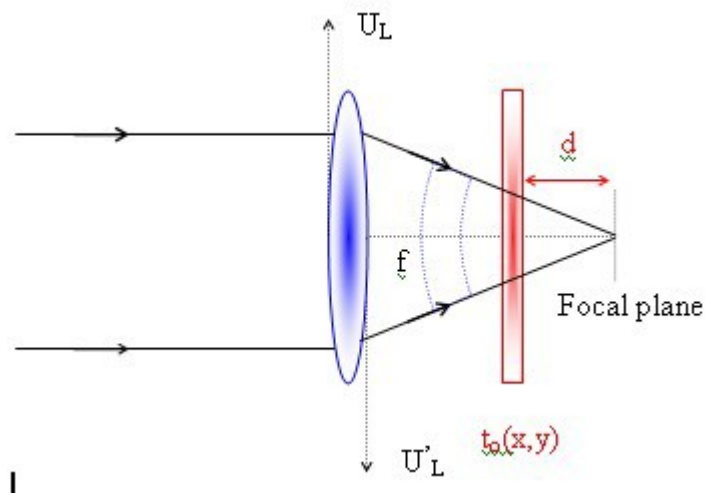


Figure IV-7- Object placed behind the lens at a distance d from the focal plane

Basing ourself on figure IV-8,

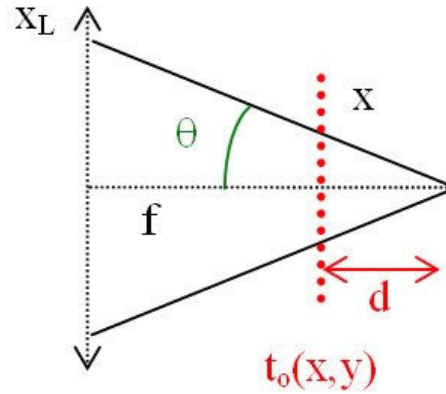


Figure IV-8- Simplified diagram of Figure IV-7. Definition of the variables involved in the calculation.

A simple calculation done in the geometric optics framework with the paraxial approximation shows that:

- the amplitude at the object location is: $A' \approx Af/d$.
- the finite extension of the illuminated area, intersection between the object plane and the light cone can be represented by the effective pupil function: $P(x_L, y_L) = P(xf/d, yf/d)$,
- the spherical wave illuminating the object can be written:

$$\exp(-jkr) \approx \exp(-jkd) \cdot \exp[-jk(x^2 + y^2)/2d]$$

Using those conditions, the complex amplitude $U_o(x, y)$ of the field transmitted by the object can be written:

$$U_o(x, y) = \left\{ \frac{Af}{d} P\left(\frac{xf}{d}, \frac{yf}{d}\right) \cdot e^{-j\frac{k}{2d}(x^2+y^2)} \right\} \cdot t_o(x, y)$$

By applying the Fresnel diffraction formula (Eq. III-2) to go from the object plane to the focal plane ($z = d$), we find an expression of the amplitude in that latter plane:

$$\tilde{U}_f(u', v') = \frac{e^{jkd}}{j\lambda d} e^{j\frac{k}{2d}[u'^2+v'^2]} \int_{-\infty}^{\infty} \int U_o(x, y) e^{j\frac{k}{2d}[x^2+y^2]} \cdot e^{-j\frac{2\pi}{\lambda d}[xu'+yv']} dx dy$$

By replacing $U_o(x, y)$ (Eq. IV-7) by its value, we notice that the quadratic phase term inside the integral vanishes.

$$\tilde{U}_f(u', v') = \frac{A}{j\lambda d} e^{j\frac{k}{2d}[u'^2+v'^2]} \cdot \frac{f}{d} \cdot \int_{-\infty}^{\infty} \int t_o(x, y) P\left(\frac{xf}{d}, \frac{yf}{d}\right) \cdot e^{-j\frac{2\pi}{\lambda d}[xu'+yv']} dx dy$$

Except for the quadratic phase term in front of the integral, the amplitude distribution at the focal plane is therefore the *FT* of the object part which is limited by the projection of the lens aperture at the object plane. This result is the same as the one obtained in the case where the object is located right near the lens. However, here, the experimentalist can easily modify the *FT* scale.

By increasing d , he will increase the *FT* dimensions until the object gets very near to the lens

$$(d = f)^u = \frac{u'}{\lambda d} \Rightarrow u' = \lambda du$$

. This property can be useful in spatial filtering applications.

4.3. Image formation under monochromatic illumination

We consider an object located at a distance d_o in front of the lens; at a distance d_i (in the image plane) we have a field distribution $U_i(x_i, y_i)$ (see figure IV-9).

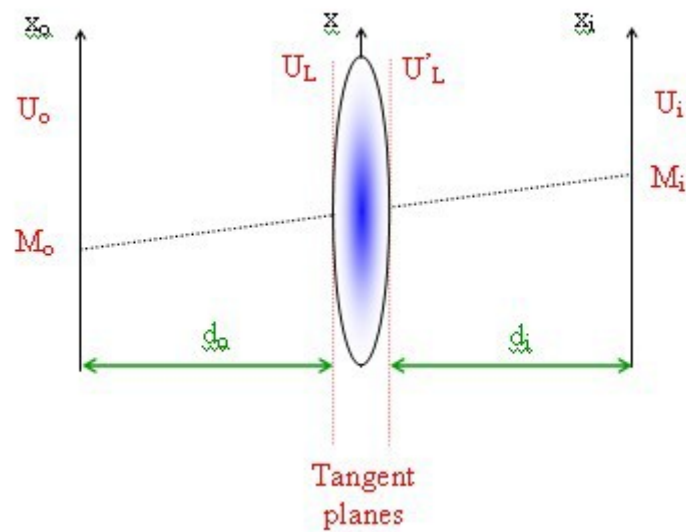


Figure IV-9- Formation of images in monochromatic lighting

We define the magnification $G_s = d_i/d_o$. We assume that :

$$1/f = 1/d_o + 1/d_i$$

(the Gaussian lens formula is satisfied).

We first consider the image predicted by geometric optics. It is simply a reversed and dilated version of the object:

$$U_i(x_i, y_i) = U_o(-x_i/G_s, -y_i/G_s)/G_s$$

On the wave optics point of view, and taking into account the linear property of the propagation phenomenon, the expression of U_i is a superposition integral (Huygens-Fresnel principle):

$$U_i(x_i, y_i) = \int_{-\infty}^{\infty} \int h(x_i, y_i ; x_o, y_o) U_o(x_o, y_o) dx_o dy_o$$

where $h(M_i, M_o)$ is the amplitude of the field produced at the point $M_i(x_i, y_i)$ by a point source of unit amplitude and located at the point $M_o(x_o, y_o)$ in the object plane.

$h(M_i, M_o)$ fully characterizes the optical properties of the system. It is called the **point-spread function of the imaging system**.

To determine h , we consider a point source in the object plane $\delta(M_o)$ (see figure IV-10).

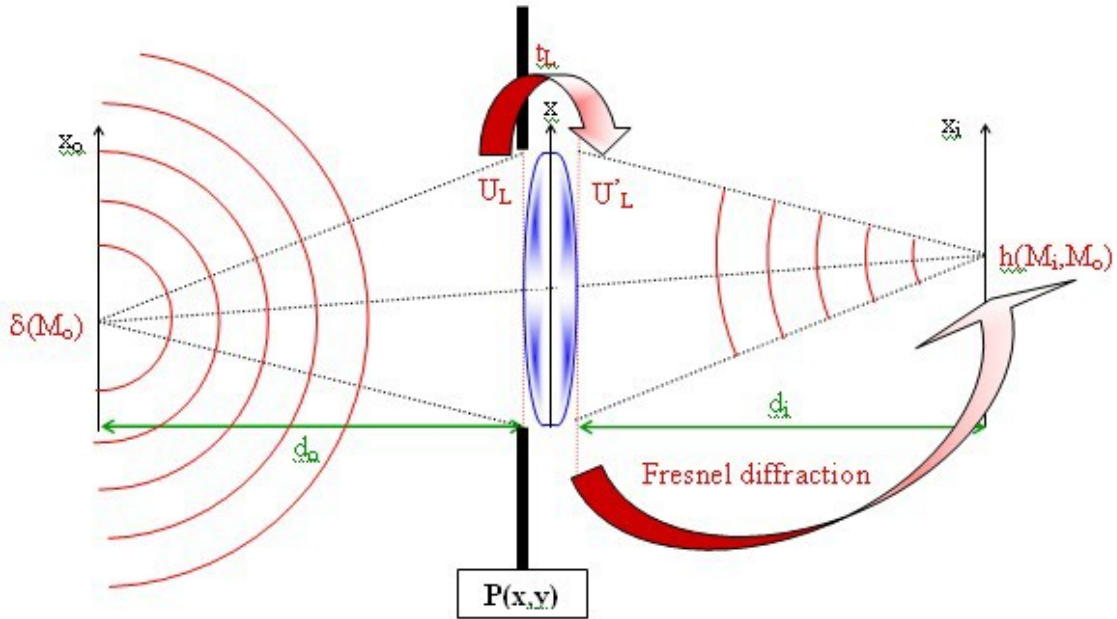


Figure IV-10- Determination of the impulse response of an imaging system

This point source emits a spherical wavelet which propagates on a distance d_o until the plane tangent to the lens. At this plane, the field amplitude is:

$$U_L(x, y) \approx \exp\{+jk[(x - x_o)^2 + (y - y_o)^2]/2d_o\}/d_o$$

This field meets the pupil function $P(x, y)$ characterizing the finite dimensions of the lens. We apply the phase transform:

$$t_L(x, y) = \exp[-jk(x^2 + y^2)/2f]$$

to this field to obtain the transmitted field U'_L . Using Fresnel diffraction law on a finite distance d_i gives the expression of the point-spread function:

$$h(M_i, M_o) = \frac{1}{\lambda^2 d_o d_i} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x, y) \cdot e^{-j\frac{2\pi}{\lambda d_i} [x(x_i + G_s x_o) + y(y_i + G_s y_o)]} dx dy$$

$h(M_i, M_o)$ represents the Fraunhofer diffracting figure of the lens aperture (the pupil) centered at point $x_i = -G_s x_o$; $y_i = -G_s y_o$.

a) Relation between object and image

We define $u = \frac{x}{\lambda d_i}$; $v = \frac{y}{\lambda d_i}$ and $x'_o = -G_s x_o$; $y'_o = -G_s y_o$. The expression (IV-9) becomes:

$$h(M_i, M'_o) = G_s \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\lambda d_i u, \lambda d_i v) \cdot e^{-j2\pi [u(x_i - x_o) + v(y_i - y_o)]} du dv$$

We deduce that the point-spread function is spatially invariant, i.e.:

$$h(x_i, y_i; x'_o, y'_o) \equiv h(x_i - x'_o, y_i - y'_o)$$

Defining $h' = \frac{1}{G_s} h$, the superposition integral (IV-8) becomes:

$$U_i(x_i, y_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h'(x_i - x'_o, y_i - y'_o) \cdot \left[\frac{1}{G_s} \cdot U_o\left(-\frac{x_o}{G_s}, -\frac{y_o}{G_s}\right) \right] dx'_o dy'_o$$

The image appears as the convolution of the point-spread function with the image obtained with geometric optics.

Fondamental

$$U_i(x_i, y_i) = h'(x_i, y_i) \otimes U_g(x_i, y_i)$$

$$U_g(x_i, y_i) = \frac{1}{G_s} \cdot U_o\left(-\frac{x_i}{G_s}, -\frac{y_i}{G_s}\right)$$

$$h'(x_i, y_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\lambda d_i u, \lambda d_i v) \cdot e^{-j2\pi[x_i u + y_i v]} du dv$$

This result shows that when we take into account the effect of diffraction, the image is no longer the exact replica of the object. It is attenuated by the convolution with the system point spread function, which affects mostly the fine details in the object.

5. General study of the image formation systems

5.1. General scheme

We consider an optical system composed of several lenses (converging and diverging). This system is described as a "black box" characterized by its terminal elements : entrance pupil (finite aperture) and exit pupil (see figure V-1).

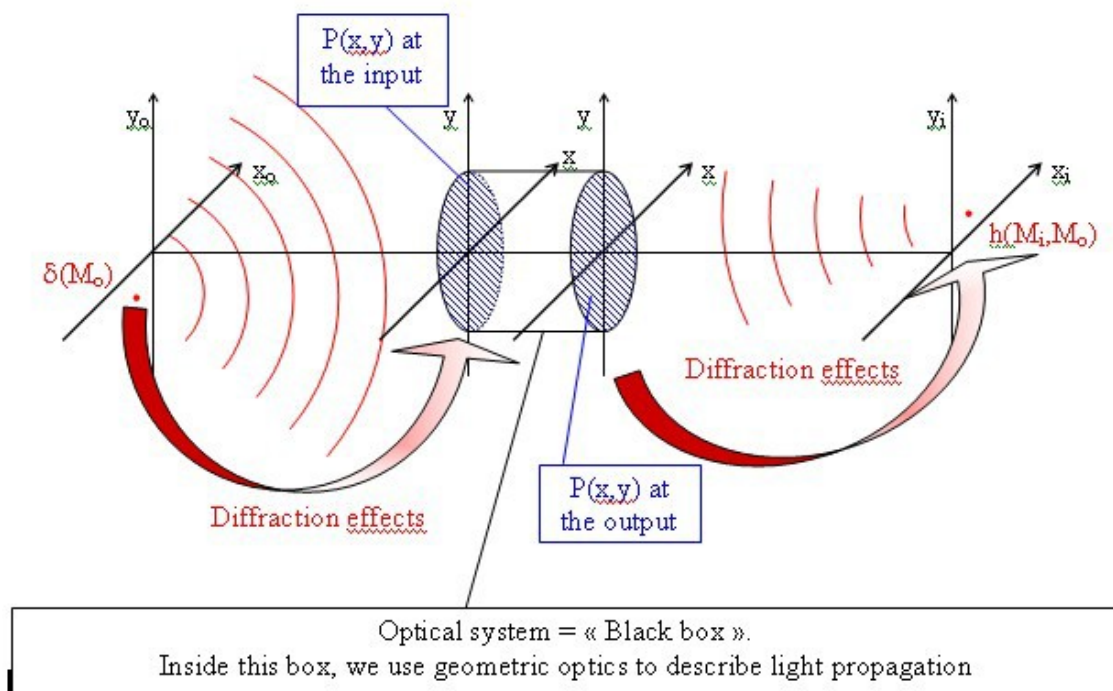


Figure V-1- General diagram of an image-forming system

The light transmission between the entrance and exit planes is described by geometric optics. The pupils are determined by forming the geometric images of the smallest aperture of the system through all the elements that precede and follow that aperture. The two points of view that consist in considering that the image resolution is limited by the finite extension of the entrance pupil or the exit pupil are completely equivalent. Indeed, one pupil is simply the geometric image of the other pupil (by definition).

In 1873, Abbe showed that only a part of the spectral components of an object can go through the entrance pupil, because of the pupil finite extension. The components stopped by the aperture are the high frequency components of the object (see figure V-2).

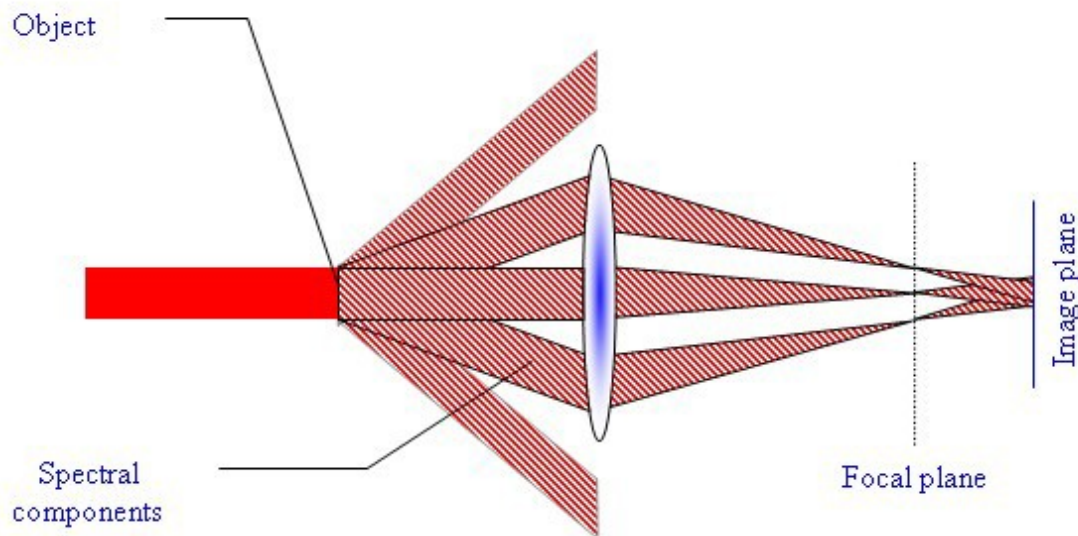


Figure V-2 - Abbe imaging system

5.2. Case of a coherent illumination

Transfer function

We showed that (Eq. IV-10), $U_i(x_i, y_i) = h'(x_i, y_i) \otimes U_g(x_i, y_i)$, i.e :

$$U_i(x_i, y_i) = \int_{-\infty}^{\infty} \int h'(x_i - x'_o, y_i - y'_o) \cdot U_g(x'_o, y'_o) \cdot dx'_o dy'_o$$

We define:

- $\tilde{G}_g(u, v) = FT[U_g(x'_o, y'_o)]$ the spectrum of the input field;
- $\tilde{G}_i(u, v) = FT[U_i(x_i, y_i)]$ the spectrum of the output field;
- $\tilde{H}(u, v) = FT[h'(x_i, y_i)]$ the point spread function spectrum.

By applying the convolution theorem:

$$\tilde{G}_i(u, v) = \tilde{H}(u, v) \cdot \tilde{G}_g(u, v)$$

In what follows, the function \tilde{H} will be called coherent transfer function.

Remarque

We note that:

$$\tilde{H} = FT[h'] = FT\left(TF(P(\lambda d_i u, \lambda d_i v))\right)$$

therefore: $\tilde{H}(u, v) = P(-\lambda d_i u, -\lambda d_i v)$

- The pupil function P only takes 0 and 1 values: the same is true for the coherent transfer function.
- There is a finite band in the frequency domain where the diffraction-limited system transmits frequency components without distorting their phase and amplitude. Outside of this band, the frequency response vanishes rapidly.
- By choosing a coordinate system such that: $x_i = -x_i$; $y_i = -y_i$, we can write:

$$H(u, v) = P(\lambda d_i u, \lambda d_i v)$$

5.3. Case of an incoherent illumination

a) Introduction

In the case of a (spatially) coherent illumination (see figure V-3), the complex amplitudes at all object points follow the same evolution with time.

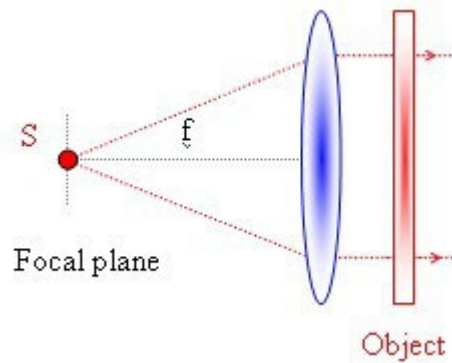


Figure V-3- General diagram of coherent lighting.

Experimentally, this is realized by placing a light source of small dimension at the front focal plane of a lens.

The illumination is said to be spatially incoherent if the source is (spatially) wide enough (see figure V-4).

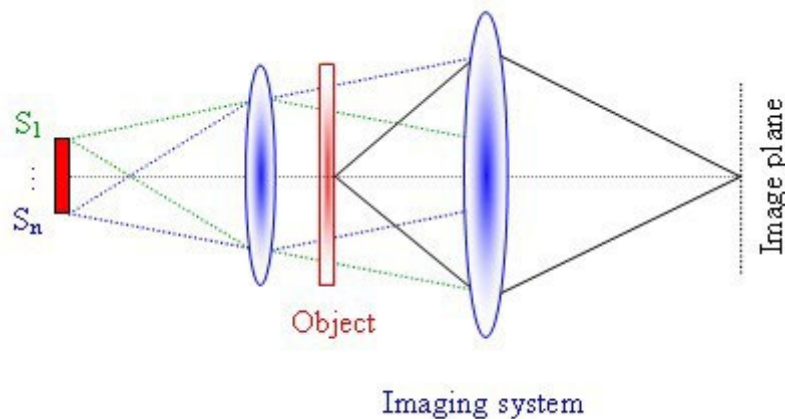


Figure V-4- General diagram of incoherent lighting.

The complex amplitudes of all points forming the source vary on a random fashion, and independently from one another. There is no fixed phase relationship between a wave train emitted by S_1 and another one emitted by S_n and the temporal average over the interference term is equal to zero.

$$\langle I \rangle = \langle I_1 \rangle + \langle I_n \rangle + \langle 2\sqrt{I_1 I_n} \cos(\phi_1(t) - \phi_2(t)) \rangle$$

$$\langle I \rangle = \langle I_1 \rangle + \langle I_n \rangle$$

In the case of an incoherent illumination, we have to sum the intensities, and not the fields.

The image is therefore written as a linear transformation of the intensity [[Introduction to Fourier Optics]] :

$$I_i(x_i, y_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h'(x_i - x'_o, y_i - y'_o)|^2 \cdot I_g(x'_o, y'_o) \cdot dx'_o dy'_o$$

The image intensity is a convolution between the ideal intensity (obtained with geometric optics) and a point spread function proportional to the square modulus of the point spread function obtained with coherent illumination.

b) Frequency response of an optical system (incoherent illumination)

We define the normalized spectra of I_g and I_i :

$$\tilde{G}_g(u, v) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_g(x'_o, y'_o) \cdot e^{-j2\pi(ux'_o + vy'_o)} dx'_o dy'_o}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_g(x'_o, y'_o) \cdot dx'_o dy'_o}$$

$$\tilde{G}_i(u, v) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_i(x_i, y_i) \cdot e^{-j2\pi(ux_i + vy_i)} dx_i dy_i}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_i(x_i, y_i) \cdot dx_i dy_i}$$

The spectra are normalized with respect to the integral value at zero frequency, which represents the continuous component or background of the image.

In a similar fashion, we define the normalized transfer function of the system:

$$\tilde{H}_i(u, v) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h'(x_i, y_i)|^2 \cdot e^{-j2\pi(ux_i + vy_i)} dx_i dy_i}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h'(x_i, y_i)|^2 \cdot dx_i dy_i}$$

By applying the convolution theorem to Eq. V-2, we obtain:

$$\tilde{G}_i(u, v) = \tilde{H}_i(u, v) \cdot \tilde{G}_g(u, v)$$

\tilde{H}_i is called «**optical transfer function**» («OTF») or «**modulation transfer function**» («MTF»).

c) Relation between the MTF and the coherent transfer function

$$\tilde{H}(u, v) = TF(h') \text{ et } \tilde{H}_i(u, v) = \frac{TF(|h'|^2)}{TF(|h'|^2)_{(u=0, v=0)}}$$

Using the autocorrelation theorem:

$$\tilde{H}_i(u, v) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{H}(u', v') \tilde{H}^*(u' - u, v' - v) \cdot du' dv'}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{H}(u', v')|^2 \cdot du' dv'}$$

By applying the change of variables $u'' = u' - u/2$ and $v'' = v' - v/2$, we obtain a symmetrical expression:

$$\tilde{H}_i(u, v) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{H}(u'' + \frac{u}{2}, v'' + \frac{v}{2}) \tilde{H}^*(u'' - \frac{u}{2}, v'' - \frac{v}{2}) \cdot du'' dv''}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{H}(u'', v'')|^2 \cdot du'' dv''}$$

For a coherent system, we have $\tilde{H}(u, v) = P(\lambda d_i u, \lambda d_i v)$, and therefore we obtain:

$$\tilde{H}_i(u, v) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u' + \frac{\lambda d_i u}{2}, v' + \frac{\lambda d_i v}{2}) P^*(u' - \frac{\lambda d_i u}{2}, v' - \frac{\lambda d_i v}{2}) \cdot du' dv'}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u', v') \cdot du' dv'}$$

where we replaced u'' with u' and P^2 by P in the denominator (since $P = 1$ or 0). The relation (V-3) serves as a fundamental link between coherent and incoherent systems.

In the expression of \tilde{H}_i appears the area of the common surface between two identical pupil functions:

- one centered at the point of coordinates : $(\frac{\lambda d_i u}{2}, \frac{\lambda d_i v}{2})$,
- the other centered at the diametrically opposed point, of coordinates : $(-\frac{\lambda d_i u}{2}, -\frac{\lambda d_i v}{2})$.

The denominator normalizes this area with the total pupil area (see figure V-5).

$$\tilde{H}_i(u,v) = \frac{\text{Overlapping surface area}}{\text{Total surface area}}$$

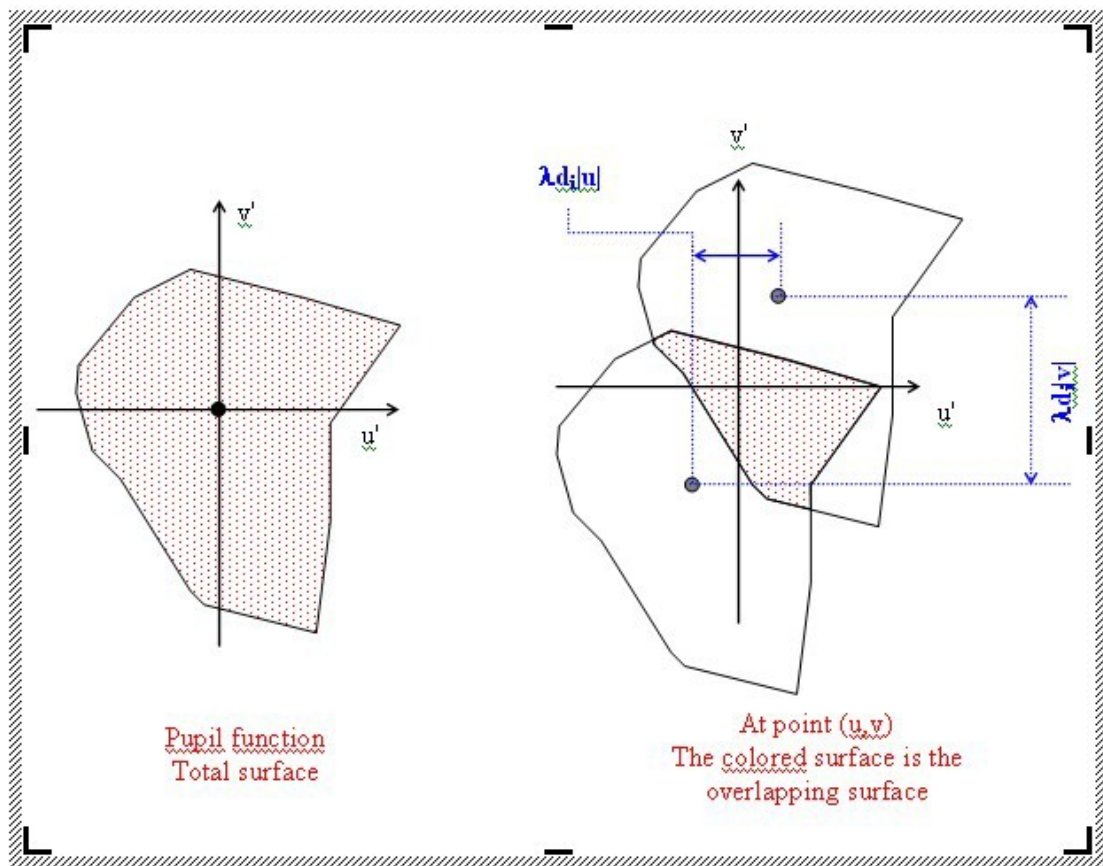


Figure V-5 - Pupillary function and MTF calculation in the case of incoherent lighting.

For more details pertaining to this subject, the reader can refer to [[Introduction to Fourier Optics]] and references therein.

Fondamental

Summary

A system with coherent illumination is linear in amplitude. The intensity is given by
 $I_i = |h * U_g|^2$.

A system with incoherent illumination is linear in intensity. The intensity is given by
 $I_i = |h|^2 * |U_g|^2$.

The frequency spectrum of the intensity can be written:

- **in the coherent illumination regime:** $TF(I_i) = (\tilde{H} \cdot \tilde{G}_g) \otimes (\tilde{H} \cdot \tilde{G}_g)$.

- **in the incoherent illumination regime:** $TF(I_i) = (\tilde{H} \otimes \tilde{H}) \cdot (\tilde{G}_g \otimes \tilde{G}_g)$.

\tilde{G}_g is the spectrum of U_g , \tilde{H} is the coherent transfer function.

* **corresponds to the convolution operator:** $g * h = \int_{-\infty}^{\infty} \int g(u, v)h(x - u, y - v)du dv$.

\otimes **corresponds to the autocorrelation:**

$$F(u, v) \otimes F(u, v) = \int_{-\infty}^{\infty} \int F(u', v')F^*(u' - u, v' - v) \cdot du' dv'$$

III. Case study

1. Fraunhofer diffraction of a sinusoidal amplitude grating

Until now, the aperture transmittance $t(x, y)$ was a binary function (taking 0 values outside the aperture and 1 values inside). However, we can introduce absorbing spatial distributions (for example using a photographic film) and thereby obtain all transmittance values ranging between 0 and 1.

We consider a sinusoidal amplitude grating limited by a square aperture of side length L :

$$t(x_1, y_1) = \left[\frac{1}{2} + \frac{m}{2} \cos(2\pi f_0 x_1) \right] \text{rect}\left(\frac{x_1}{L}\right) \text{rect}\left(\frac{y_1}{L}\right)$$

where f_0 is the spatial frequency of the grating and m represents the peak-to-peak variation of the amplitude. m is called the amplitude modulation depth (see figure EC1(a)).

Figure CS1(b) shows the grating image. Of course, the grating step has been voluntarily increased in this figure to visualize the sinusoidal transmittance.

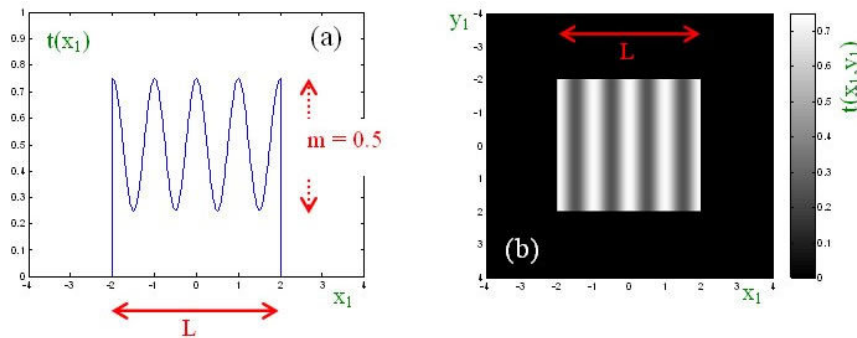


Figure EC1 - (a) Evolution of the transmittance function of a sinusoidal network in amplitude; (b) image of this network limited by a square opening.

If the screen is illuminated under a normal incidence with a monochromatic plane wave of unit amplitude, the field distribution at the aperture is simply equal to $t(x_1, y_1)$.

The Fraunhofer diffraction figure can be determined through the calculation of $FT(t)$:

$$FT[t(x_1, y_1)] = FT\left[\frac{1}{2} + \frac{m}{2} \cos(2\pi f_0 x_1)\right] * FT\left[\text{rect}\left(\frac{x_1}{L}\right) \text{rect}\left(\frac{y_1}{L}\right)\right]$$

We define:

$$\tilde{A} = FT\left[\frac{1}{2} + \frac{m}{2} \cos(2\pi f_0 x_1)\right] = \frac{1}{2} \delta(u, v) + \frac{m}{4} \delta(u + f_0, v) + \frac{m}{4} \delta(u - f_0, v)$$

$$\tilde{B} = FT\left[\text{rect}\left(\frac{x_1}{L}\right) \text{rect}\left(\frac{y_1}{L}\right)\right] = L^2 \text{sinc}(Lu) \text{sinc}(Lv)$$

Using the distributivity property of the convolution product, we can write:

$$\begin{aligned} \tilde{A} * \tilde{B} &= \frac{L^2}{2} \text{sinc}(Lu) \text{sinc}(Lv) * \delta(u, v) + m \frac{L^2}{4} \text{sinc}(Lu) \text{sinc}(Lv) * \delta(u + f_0, v) + \\ & m \frac{L^2}{4} \text{sinc}(Lu) \text{sinc}(Lv) * \delta(u - f_0, v) \end{aligned}$$

Considering that translating a function $f(x, y)$ of a quantity a is equivalent to taking its convolution with the translated Dirac delta function ($\delta(x - a)$) :

$$f(x, y) * \delta(x - a, y) = \int \int f(u, v) \delta(x - a - u, y - v) du dv = f(x - a, y)$$

we obtain:

$$TF[t(x_1, y_1)] = \frac{L^2}{2} \text{sinc}(Lv) \left\{ \text{sinc}(Lu) + \frac{m}{2} \text{sinc}[L(u + f_0)] + \frac{m}{2} \text{sinc}[L(v - f_0)] \right\}$$

We can deduce the field distribution in amplitude:

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x_0^2 + y_0^2)} \cdot TF[t(x_1, y_1)] \begin{cases} u = \frac{x_0}{\lambda z} \\ v = \frac{y_0}{\lambda z} \end{cases}$$

In figure CS2(a), we represented the 3 sinc functions centered at $-f_0$, 0 and $+f_0$ in the case where $m = 1$.

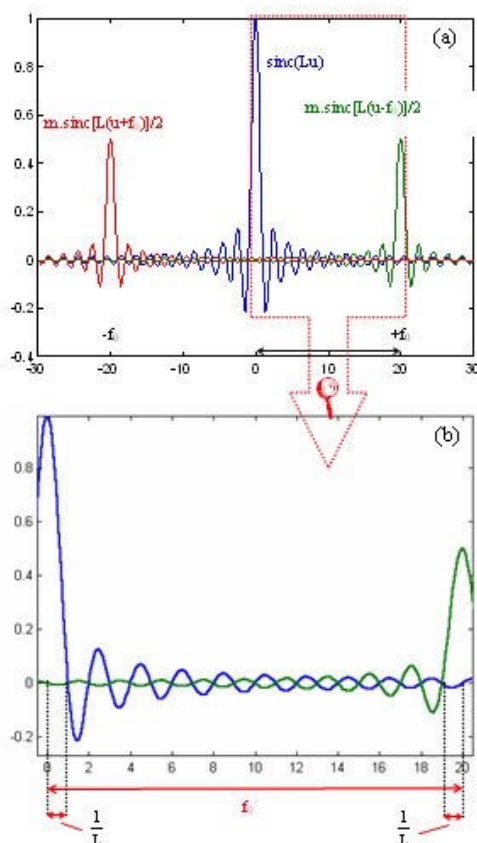


Figure EC2 - (a) the 3 cardinal sine functions appearing in the amplitude of the diffracted field; (b) enlargement of the previous figure to compare f_0 to $2/L$.

An enlargement of this figure is shown on figure EC2(b) where we can see the central peak widths of the two sinc functions centered at 0 and $+f_0$.

If $f_0 \gg \frac{2}{L}$, the product of those two functions can be neglected, because when one function reaches its maximum the other takes values close to zero. We can therefore neglect the overlap between the 3 sinc functions. When calculating the intensity, the cross-products between these functions are negligible compared to the square of each of these functions.

Finally, the intensity in the observation plane becomes:

$$I(x_0, y_0) = \left(\frac{L^2}{2\lambda z}\right)^2 \text{sinc}^2\left(\frac{Ly_0}{\lambda z}\right) \times \left\{ \text{sinc}^2\left(\frac{Lx_0}{\lambda z}\right) + \frac{m^2}{4} \text{sinc}^2\left[\frac{L}{\lambda z}(x_0 + f_0\lambda z)\right] + \frac{m^2}{4} \text{sinc}^2\left[\frac{L}{\lambda z}(x_0 - f_0\lambda z)\right] \right\}$$

An image of the diffraction figure obtained using this latter equation is shown on figure CS3(a).

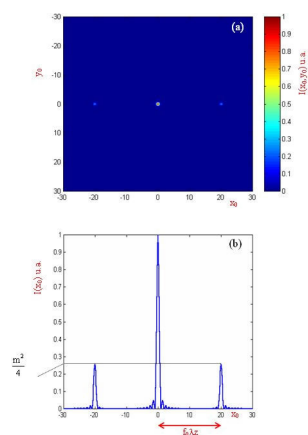


Figure EC3 - (a) Image showing the Fraunhofer diffraction pattern of the grating; (b) Evolution of the normalized intensity distribution (at $y_0 = 0$) when $m = 1$.

A part of the light is deflected in two lateral lobes. The central part is called zero-order component and the two lateral lobes are called 1st order components. Figure CS3(b) represents the evolution of the normalized intensity for $y_0 = 0$ and $m = 1$. We can note that the distance between the first order diffraction lobes and the central lobe is $f_0 \lambda z$. In addition, in the best case (when $m = 1$), the intensity in the diffracted lobes cannot exceed 0.25 for a grating sinusoidal in amplitude.

IV. Exercises

1. Knowledge test

Image formation

Answer the following questions

Question 1

[Solution n°1 p 37]

Calculate the Fraunhofer diffraction figure of a rectangular aperture and a circular aperture. Compare the central lobe widths when the square side length is equal to the circular aperture diameter. Conclude.

Question 2

[Solution n°2 p 42]

We consider two imaging systems: one with a square pupil of side length a , and one with a circular pupil of diameter a . For each system, calculate the coherent transfer function relative to those pupils and determine the cutoff frequencies as a function of a , the wavelength λ , and the distance d_i between the pupil and the image plane.

Question 3

[Solution n°3 p 43]

Calculate the transfer function in the case of an incoherent illumination for a square pupil of side length a , and determine the cutoff frequency as a function of a , the wavelength λ , and the distance d_i between the pupil and the image plane. Compare this result with the cutoff frequency obtained in the previous question.

Solution des exercices

>Solution n°1 (exercice p. 36)

Examples of Fraunhofer diffraction figures - Rectangular aperture

We consider a rectangular aperture of amplitude transmittance t :

$$t(x_1, y_1) = \text{rect}\left(\frac{x_1}{L_x}\right) \cdot \text{rect}\left(\frac{y_1}{L_y}\right)$$

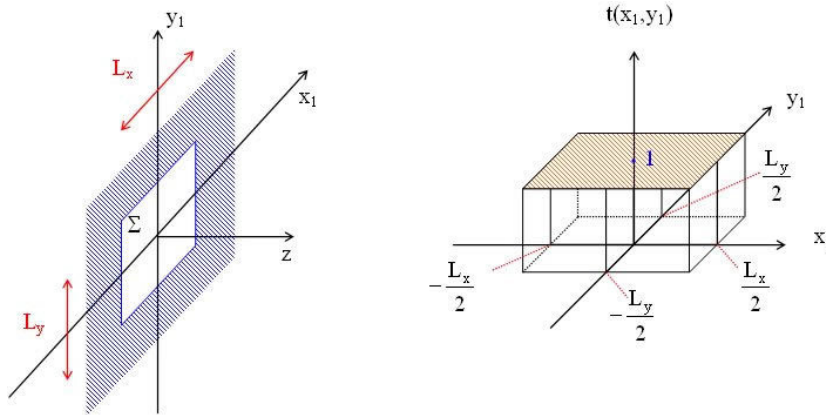


Figure TC1- Rectangular aperture and the representation of its transmittance.

L_x and L_y are the aperture dimensions along x_1 and y_1 .

If the aperture is illuminated under normal incidence by a monochromatic plane wave of unit amplitude ($U_i(x_1, y_1) = 1$), the field distribution at the aperture ($U_t(x_1, y_1)$) is equal to the transmittance function:

$$t(x_1, y_1) = \frac{U_t(x_1, y_1)}{U_i(x_1, y_1)} = U_t(x_1, y_1)$$

Using the expression (III-3) characterizing the field distribution in the Fraunhofer diffraction figure, we can write:

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x_0^2 + y_0^2)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} t(x_1, y_1) e^{-j\frac{2\pi}{\lambda z}[x_0 x_1 + y_0 y_1]} dx_1 dy_1$$

where the integral can be identified as the $FT[t(x_1, y_1)] = \tilde{T}(u, v)$ at the point $u = x_0/\lambda z$; $v = y_0/\lambda z$. Using table 2 (section "2-dimensional Fourier analysis", paragraph "Couples of transforms relating to several separable functions in Cartesian coordinates"), we easily obtain:

$$\tilde{T}(u, v) = L_x L_y \text{sinc}(L_x u) \cdot \text{sinc}(L_y v)$$

As a result, the expression of the field at the observation plane becomes:

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x_0^2 + y_0^2)} L_x L_y \text{sinc}\left(L_x \frac{x_0}{\lambda z}\right) \cdot \text{sinc}\left(L_y \frac{y_0}{\lambda z}\right)$$

and the diffracted light intensity can be written:

$$I(x_0, y_0) = \left(\frac{L_x L_y}{\lambda z}\right)^2 \cdot \text{sinc}^2\left(\frac{L_x x_0}{\lambda z}\right) \cdot \text{sinc}^2\left(\frac{L_y y_0}{\lambda z}\right)$$

The image aspect in the observation plane (in false colors and with $2L_x = L_y$) is shown on the following figure:

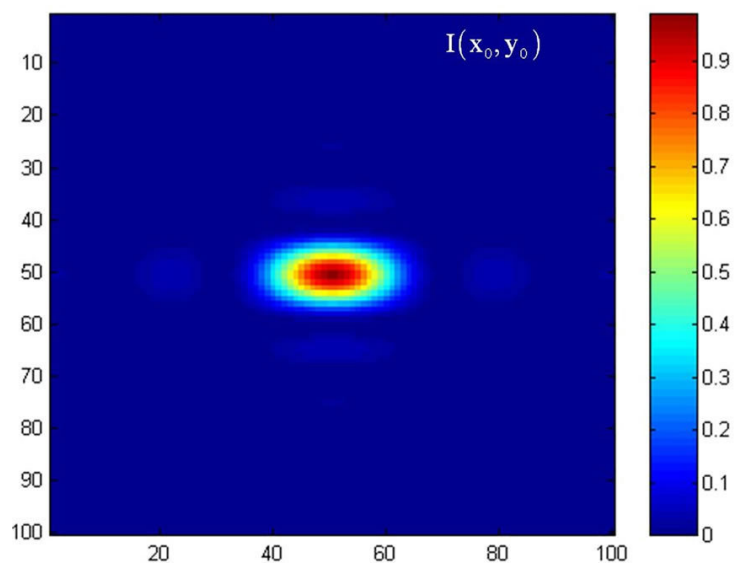


Figure TC2 - Image representing the Fraunhofer diffraction pattern of a rectangular aperture. A 3D plot of the same function is shown on figure KT3.

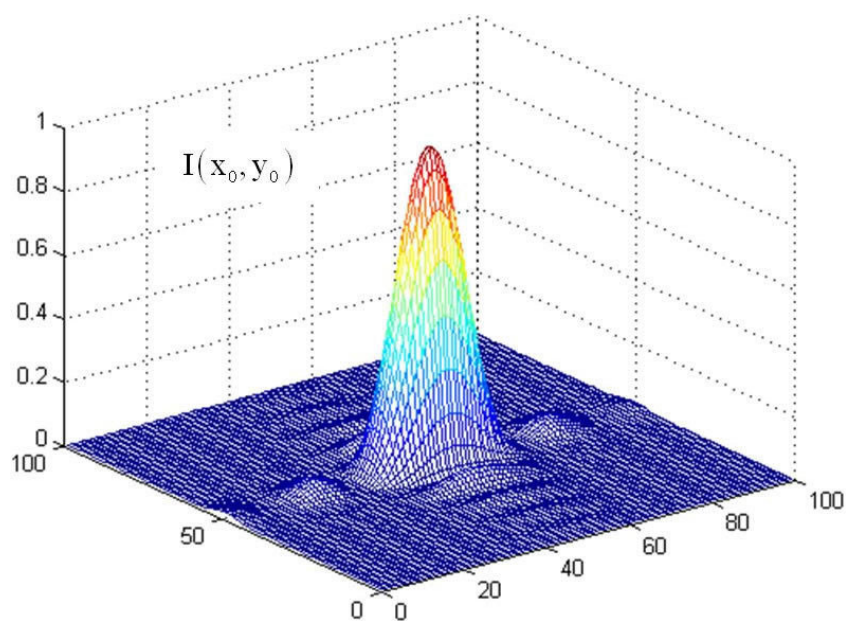


Figure TC3 - Evolution of the intensity in the observation plane of the image in figure TC2

One can note the presence of small diffraction lobes which are not centered on the axes x_0 et y_0 ; those lobes are invisible in the two previous figure because of their small intensity. To visualize those lobes, we represented the intensity logarithm on figure KT4:

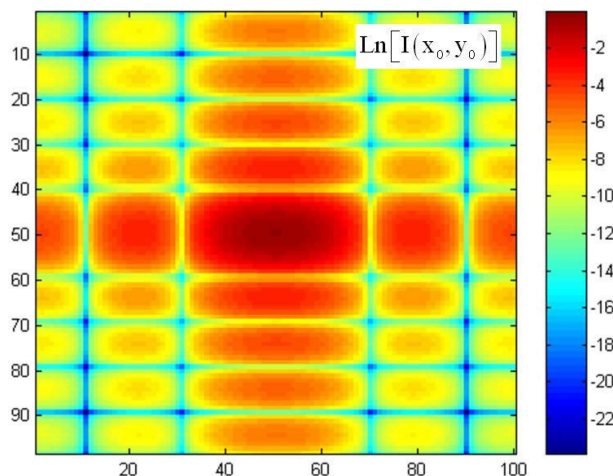


Figure TC4 - Image representing the log of the intensity obtained in figure TC2.

We note that the sinc function (defined in table 1 in the section "Two-Dimensional Fourier Analysis") takes zero values for integer arguments. The first zeros obtained on both sides of the origin are given by:

$$x_0 = \pm \frac{\lambda z}{L_x}$$

The central lobe width along the x_0 axis in the observation plane is therefore:

$$D_{rect} = 2 \frac{\lambda z}{L_x}$$

Examples of Fraunhofer diffraction figures – Circular aperture

We consider a circular aperture of radius R (see figure KT5).

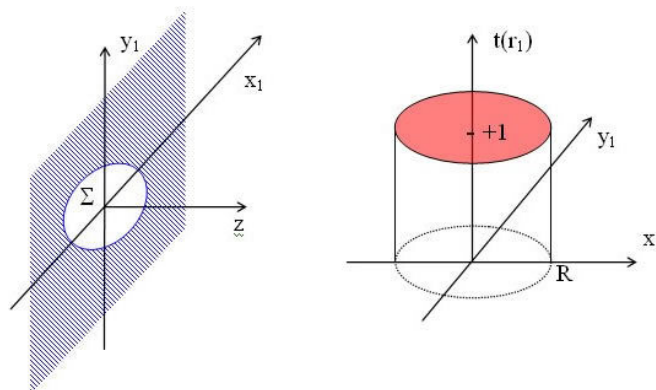


Figure TC5- Ouverture circulaire et la représentation de sa transmittance.

We define r_1 and r_0 as the radial coordinates at the aperture plane and at the observation plane (respectively). Then:

$$t(r_1) = \text{circ}\left(\frac{r_1}{R}\right)$$

If the aperture is illuminated under normal incidence with a monochromatic plane wave of unit amplitude ($U_i(r_1) = 1$), the field distribution at the aperture ($U_t(r_1)$) is equal to the transmittance function:

$$t(r_1) = \frac{U_t(r_1)}{U_i(r_1)} = U_t(r_1)$$

By using expression (III-3) which characterizes the field distribution in the Fraunhofer diffraction pattern, we can write:

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}[x_0^2+y_0^2]} \int_{-\infty}^{+\infty} \int t(x_1, y_1) e^{-j\frac{2\pi}{\lambda z}[x_0x_1+y_0y_1]} dx_1 dy_1$$

where we recognize under the integral sign:

$$TF[t(r_1)] = TF\left[\text{circ}\left(\frac{r_1}{R}\right)\right]$$

$$\tilde{T}(\rho) = B\left\{\text{circ}\left(\frac{r_1}{R}\right)\right\}$$

Here, we have:

$$u = \frac{x_0}{\lambda z}$$

$$v = \frac{y_0}{\lambda z} \Rightarrow \rho = \sqrt{u^2 + v^2} = \frac{\sqrt{x_0^2 + y_0^2}}{\lambda z}$$

therefore:

$$\rho = \frac{r_0}{\lambda z}$$

We recall a result demonstrated in the paragraph I.4.4 :

$$FT[\text{circ}(r)] = J_1(2\pi\rho)/\rho$$

and by applying the scaling theorem, we easily obtain that:

$$\tilde{T}(\rho) = B\left\{\text{circ}\left(\frac{r_1}{R}\right)\right\} = R^2 \frac{J_1(2\pi R\rho)}{R\rho}$$

We multiply the numerator and the denominator by π and we note $S = \pi R^2$ the aperture surface. The distribution of the field amplitude at the observation plane becomes:

$$U(r_0) = \frac{e^{jkz}}{j\lambda z} \cdot e^{j\frac{kr_0^2}{2z}} \cdot S \cdot \frac{J_1\left(\frac{2\pi Rr_0}{\lambda z}\right)}{\frac{\pi Rr_0}{\lambda z}} \times \frac{2}{2}$$

$$U(r_0) = \frac{e^{jkz}}{j\lambda z} \cdot e^{j\frac{kr_0^2}{2z}} \cdot S \cdot \frac{2 \cdot J_1(kRr_0/z)}{kRr_0/z}$$

The intensity can therefore be written:

$$I(r_0) = \left(\frac{S}{\lambda z}\right)^2 \cdot \left[\frac{2 \cdot J_1(kRr_0/z)}{kRr_0/z}\right]^2$$

This intensity distribution is called «**Airy pattern**». By normalizing the radial variable : $r = 2Rr_0/\lambda z$, we obtain a simplified expression of the diffraction pattern intensity:

$$I(r_0) = I(0)[2J_1(\pi r)/\pi r]^2$$

This intensity is shown on figure KT6.

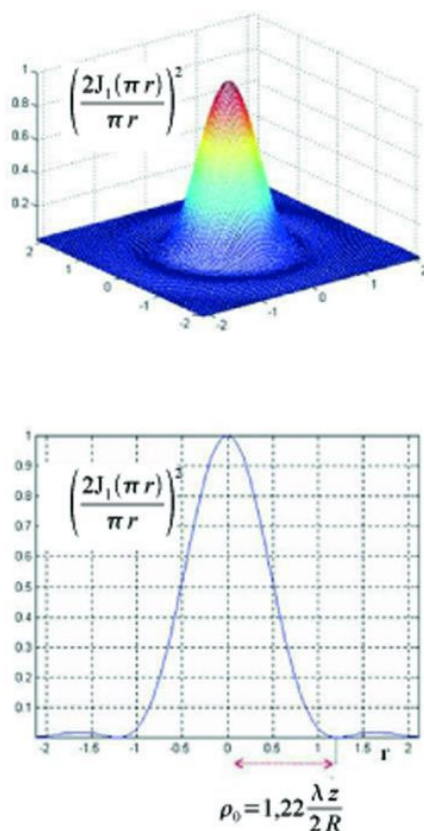


Figure TC6 - Fraunhofer diffraction by a circular aperture of radius R (Variation of the diffraction pattern (called Airy spot) in the plane and along a radial axis).

We note that the diameter of the central lobe is equal to:

$$D_{\text{circ}} = 2\rho_0 = 1.22 \frac{\lambda z}{R}$$

We represented on figure KT7 the diffraction pattern at the observation plane, in false colors.

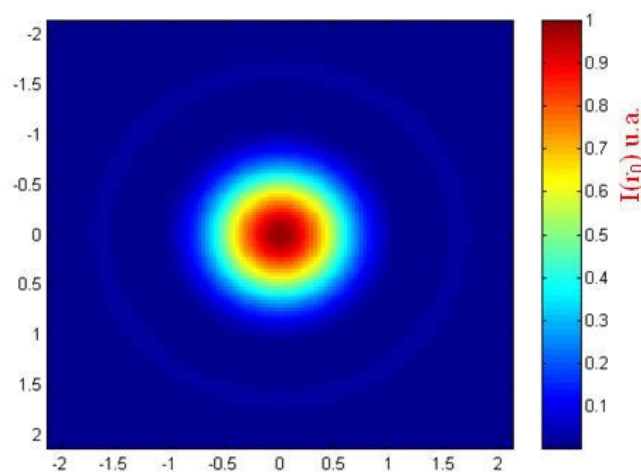


Figure TC7 - Appearance of the diffraction pattern in the observation plane (in false color).

Remarque

By comparing the central lobe width of the diffraction pattern obtained with a rectangular aperture (of width $\phi = L_x$) and with a circular aperture (of diameter $\phi = 2R$), we notice only a small difference:

$$D_{circ} = 1.22 \frac{\lambda z}{\phi/2} = 2.44 \frac{\lambda z}{\phi}$$

and

$$D_{rect} = 2 \frac{\lambda z}{\phi}$$

It is convenient in some cases (see figure KT8) where the precision is not too important to replace the circular aperture with a square aperture of side length equal to the circle diameter. This simplifies the calculations.

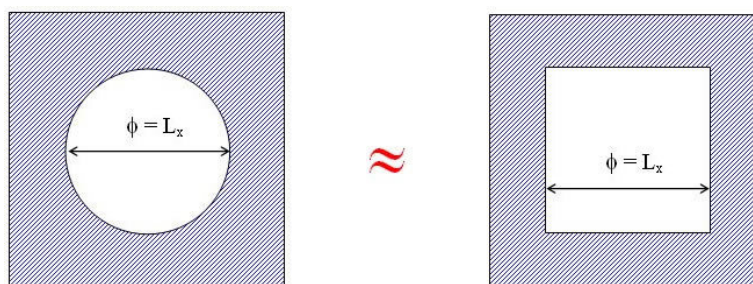


Figure TC8- Equivalence between a circular opening of diameter equal to the side of a square opening

>Solution n°2 (exercice p. 36)

Examples of transfer functions with coherent illumination

We consider a system with a square entrance (or exit) pupil of side length a , and another system with a circular pupil of diameter a .

$$P_1(x, y) = \text{rect}\left(\frac{x}{a}\right) \cdot \text{rect}\left(\frac{y}{a}\right) ; P_2(x, y) = \text{circ}\left(\frac{\sqrt{x^2 + y^2}}{a/2}\right)$$

The expression (V-1) instantly gives the corresponding coherent transfer functions (see figure KT9):

$$\tilde{H}_1(u, v) = \text{rect}\left(\frac{\lambda d_i u}{a}\right) \cdot \text{rect}\left(\frac{\lambda d_i v}{a}\right) ; H_2(u, v) = \text{circ}\left(\lambda d_i \cdot \frac{\sqrt{u^2 + v^2}}{a/2}\right)$$

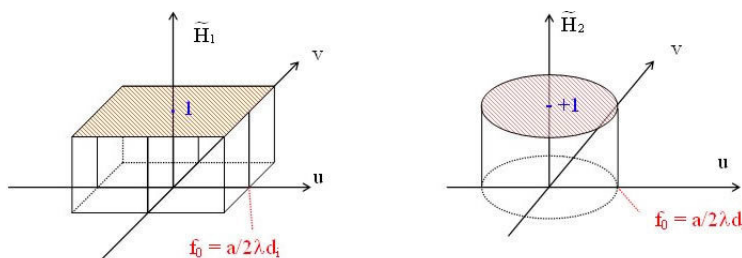


Figure TC9 - Coherent transfer function (a) square pupil (b) circular pupil

We note that in each case the cutoff frequency is $f_0 = a/2\lambda d_i$

Orders of magnitude:

- $a = 2\text{cm}, d_i = 10\text{cm}, \lambda = 10^{-4}\text{cm} \rightarrow f_0 = 100\text{periods/mm}$.

- Eye cutoff frequency : $\phi_{pupille} = 4 - 6mm, d_i = 15mm,$
 $\lambda = 0,5 \cdot 10^{-6}m \rightarrow f_0 = 300periods/mm,$

Period size:

300p. corresponds to 1mm

1p. corresponds to $1/300 = 0.3 \cdot 10^{-2}mm = 3\mu m$

This result is in good agreement with the space between cones in the retina, which is $3\mu m$.

> **Solution n°3** (exercice p. 36)

Transfer function with incoherent illumination

We consider a square pupil of side length $BB' = a$. At the point of spatial frequency (u, v) (see figure KT10), the common surface is given by: $S(u, v) = AB \times BC$. Since $AB = a - B'A$ and, for geometrical considerations, $B'A = \lambda d_i |u|$

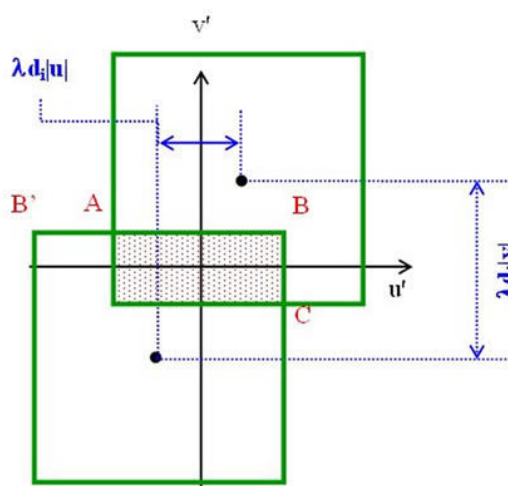


Figure TC10 - Calculation of the MTF in the case of incoherent lighting. Case of a square pupil.

we can express the common area as:

$$S(u, v) = (a - \lambda d_i |u|)(a - \lambda d_i |v|)$$

$$\text{If } \begin{cases} \lambda d_i |u| \leq a \\ \lambda d_i |v| \leq a \end{cases} \Rightarrow \begin{cases} |u| \leq \frac{a}{\lambda d_i} \\ |v| \leq \frac{a}{\lambda d_i} \end{cases}$$

$$\tilde{H}_i(u, v) = \frac{(a - \lambda d_i |u|)(a - \lambda d_i |v|)}{a^2} = \left(1 - \frac{\lambda d_i}{a} |u|\right) \left(1 - \frac{\lambda d_i}{a} |v|\right)$$

$$\text{Nothing : } tri(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{H}_i(u, v) = tri\left(\frac{u}{f_i}\right) \cdot tri\left(\frac{v}{f_i}\right)$$

$$\text{where } f_i = \frac{a}{\lambda d_i} = 2f_0$$

(f_0 is the cutoff frequency in the coherent regime.)

We note that the cutoff frequency in the incoherent regime is twice larger than the cutoff frequency in the coherent regime (but the contrast is not better). For a circular pupil, the calculation is more complex but we obtain the same result for the cutoff frequency f_i .

Bibliographie

[Aspects of diffraction theory and their application to the ionosphere] RATCLIFFE J.A., *Aspects of diffraction theory and their application to the ionosphere*, In A.C. Strickland editor, The Physical Society, London, 1956, volume XIX, Reports on Progress in Physics.

[Introduction to Fourier Optics] GOODMAN J. W., *Introduction to Fourier Optics*, Mc Graw-Hill International Editions, 1996.

[L'intégrale de Fourier et ses applications à l'optique] DUFFIEUX P.M., *L'intégrale de Fourier et ses applications à l'optique*, Faculté des sciences, Besançon, 1946.

[The Fourier integral and its application] PAPOULIS A., *The Fourier integral and its application*, Mc Graw-Hill Book Company, New York, 1962.

[The Fourier transform and its application] BRACEWELL R. N., *The Fourier transform and its application*, Mc Graw-Hill Book Company, New York, 1965.

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