

# Interferences: fundamentals

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# I.Course

The propagation of light can be interpreted in terms of rays indicating the trajectory that is followed by the light. For example, in a transparent, homogenous and isotropic medium, light propagates in a rectilinear manner. This approach is useful for describing a wide array of light phenomena. Nevertheless, light interferences cannot be described through a geometrical approach, since light also propagates in the form of waves. In 1869, the English physicist Maxwell established that such waves can be described by electromagnetic waves made up of both an electrical field  $\vec{E}$  and a magnetic field  $\vec{B}$ . The interference phenomenon occurs when the energy, the illumination, or the intensity that results from the superposition of two radiations is not equal to the sum of their respective energies, illuminations, or intensities. Historically, this result was quite surprising to the point that we refer to the paradox of interferences, which is often provocatively expressed as *light+light =darkness*.

This course examines interferences between two waves, for both plane and spherical waves. It also analyzes the effects of the source's spectral and spatial width. The case studies and exercises use the Young slits model to demonstrate how interferences between two waves are produced.

## 1. Wave description of light

### 1.1. Maxwell's formalism

In order to represent a wave, we must associate electrical  $\vec{E}$  and magnetic  $\vec{B}$  fields to physical quantities which are a function of space  $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$  and time  $t$ . Maxwell demonstrated that these fields are coupled in a given medium as described by the following equations (They are known as "Maxwell's Equations") [1 [[1]]]:

$$\left\{ \begin{array}{l} \text{div } \vec{B} = 0 \\ \text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \text{div } \vec{E} = \frac{\rho}{\epsilon} \\ \text{rot } \vec{B} = \mu \vec{J} + \epsilon \mu \frac{\partial \vec{E}}{\partial t} \end{array} \right.$$

where  $\mu$  is the permeability of the medium,  $\epsilon$  is the permittivity of the medium,  $\rho$  is associated with the free charges and the medium's polarization charges, and  $\vec{J}$  is associated with the conduction currents and the medium's polarization currents. In a vacuum, we have  $\mu_0 = 4\pi \times 10^{-7}$  and  $\epsilon_0 = 8.857 \times 10^{-12}$  and  $\epsilon_0 \mu_0 c^2 = 1$ , where  $c$  stands for the propagation speed ( $c = 3 \times 10^8$  m/s). In a linear, homogenous, and isotropic medium with no free charges we have  $\vec{J} = 0$ ,  $\rho = 0$  and  $\epsilon$  is a constant, as is the case, for example, of unconstrained glass or air.

Maxwell's equations hence produce the propagation equations for magnetic and electrical fields, which are generally expressed as:

$$\Delta \vec{E} - \frac{n^2}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

where  $\Delta$  is the Laplacian vector (we have  $\Delta = \nabla^2$ ),  $n$  is the refractive index. This equation also holds for the magnetic field.

Note that all the components of  $\vec{E}$ ,  $\{E_x, E_y, E_z\}$ , and  $\vec{B}$ ,  $\{B_x, B_y, B_z\}$  are subjected to the same propagation equation:

$$\Delta E_x - \frac{n^2}{c^2} \frac{\partial^2 E_x}{\partial t^2} = 0$$

and also for  $\{E_y, E_z, B_x, B_y, B_z\}$ .

Therefore, it is possible to synthesize the behavior of these components by a single scalar wave  $U(\vec{r}, t)$  defined by the propagation equation:

$$\Delta U(\vec{r}, t) - \frac{n^2}{c^2} \frac{\partial^2 U(\vec{r}, t)}{\partial t^2} = 0$$

where  $U(\vec{r}, t)$  represents any of these components  $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ .

A **rectilinear polarized wave** in the  $x$  direction is written:

$$\vec{E}(\vec{r}, t) = U(\vec{r}, t) \vec{e}_x$$

with:

$$\begin{cases} E_x = U(\vec{r}, t) \\ E_y = 0 \\ E_z = 0 \end{cases}$$

If the wave propagation exhibits a spherical symmetry, the change in spherical coordinates  $(r, \theta, \varphi)$  produces the following wave equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) - \frac{n^2}{c^2} \frac{\partial^2 U}{\partial t^2} = 0$$

where  $r = |\vec{r}|$ .

## 1.2. Harmonic solutions of propagation equations

The solutions of the propagation equation are either scalar or vector functions. They must reproduce the same function "farther on" and at a "later time." In order to be a solution of the equation, the function depends on the coordinate :  $t \pm \frac{n \rightarrow}{c} r$ .

Generally, for waves that propagate towards positive  $\vec{r}$  the harmonic solutions are written as follows:

$$f\left(t - \frac{n}{c} \vec{r}\right) = A(r) \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

where  $\vec{k} = k\vec{u}_0$  is the **wave vector** :

$$k = |\vec{k}| = n \frac{\omega}{c} = \frac{2\pi n}{\lambda} = \frac{2\pi}{\lambda_m}$$

and  $\vec{u}_0$  represents the vector with unit length ( $|\vec{u}_0| = 1$ ) that describes the direction of the wave propagation,  $\lambda$  is the wavelength in vacuum, and  $\lambda_m$  the wavelength in the medium. Similarly, for waves that propagate towards negative  $\vec{r}$ , the harmonic solutions are written as:

$$g\left(t + \frac{n}{c} \vec{r}\right) = A(r) \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

In the case of **plane and monochromatic waves**, the propagation is rectilinear; taking  $z$  for example, the propagation equation in vacuum would be written as:

$$\frac{\partial^2 U(\vec{r}, t)}{\partial z^2} - \frac{n^2}{c^2} \frac{\partial^2 U(\vec{r}, t)}{\partial t^2} = 0$$

and the solution that propagates towards positive  $z$  would be:

$$U(\vec{r}, t) = U(z, t) = A_0 \exp[i(kz - \varphi_0 - \omega t)]$$

### Rappel

A wave surface is defined, at a given moment, by the set of points in the space to which the wave phase,  $(kz - \varphi_0 - \omega t)$ , is identical.

The wave surface, which is associated with the propagation is given by equiphase planes such that:

$$kz - \varphi_0 - \omega t = \text{constant}$$

and it is perpendicular to the wave vector  $\vec{k}$ .

In the case of **spherical waves**, the propagation has a spherical symmetry, and the solution which propagates towards positive  $\vec{r}$  is:

$$U(\vec{r}, t) = \frac{A_0}{r} \exp[i(-\omega t + kr - \varphi_0)]$$

In this case, the component of the field depends only on  $r$  with a coordinate system centered on the point of convergence (or divergence). If the wave is divergent, this is a source point; if the wave is convergent, then it is a focalization point.

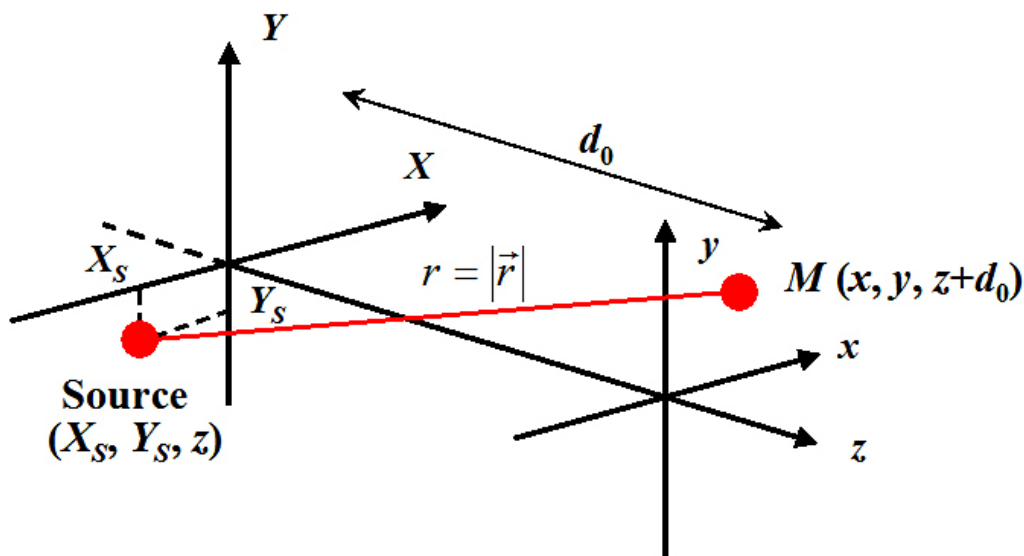
The equiphase surface that describes the wave surface is a sphere centered on the source point. A **divergent wave** is written:

$$U(\vec{r}, t) = \frac{A_0}{r} \exp(-i\omega t) \exp(ikr - \varphi_0)$$

and a **convergent wave**:

$$U(\vec{r}, t) = \frac{A_0}{r} \exp(-i\omega t) \exp(-ikr - \varphi_0)$$

It is helpful to expand the formalism in the case of a cartesian coordinate system. In figure 1, the source is situated at a point  $(X_S, Y_S, z)$  in the coordinate plane and the wave propagates by following the positives  $z$ .



We can see that:

$$r = \sqrt{d_0^2 + (x - X_S)^2 + (y - Y_S)^2}$$

The wavefront that propagates in the plane  $(x, y, z + d_0)$  situated at the distance  $(z + d_0)$  from the source is written as [1 [[1]]]:

$$U(x, y, z + d_0, t) = A_0 \exp(-i\omega t) \frac{\exp\left[\frac{2i\pi}{\lambda} \sqrt{d_0^2 + (x - X_S)^2 + (y - Y_S)^2}\right]}{\sqrt{d_0^2 + (x - X_S)^2 + (y - Y_S)^2}}$$

This expression is not easy to use. In order to reduce the complexity of writing in cartesian coordinates, we use approximations for the distance ( $d_0$ ) between the source and the observed wavefront. These approximations are based on the development of the square root in the exponential function [1 [[1]]]. Indeed, if  $a \ll 1$ , we have:

$$\sqrt{1+a} = 1 + \frac{1}{2}a - \frac{1}{8}a^2 + \frac{1}{16}a^3 + \dots$$

and if we only use the first two terms of the development, we have:

$$\sqrt{d_0^2 + (x - X_s)^2 + (y - Y_s)^2} \simeq d_0 \left( 1 + \frac{1}{2} \frac{(x - X_s)^2}{d_0^2} + \frac{1}{2} \frac{(y - Y_s)^2}{d_0^2} \right)$$

For the denominator, we use a first order approximation:

$$\sqrt{d_0^2 + (x - X_s)^2 + (y - Y_s)^2} \simeq d_0$$

Thus, from now on, the wavefront that propagates in the plane  $(x, y, z + d_0)$  situated at distance  $(z + d_0)$  from the source is written:

$$U(x, y, z + d_0, t) = \frac{A_0}{d_0} \exp(-i\omega t) \exp\left(\frac{2i\pi d_0}{\lambda}\right) \exp\left[\frac{i\pi}{\lambda d_0} \left( (x - X_s)^2 + (y - Y_s)^2 \right)\right]$$

### Remarque

The approximations (known as Fresnel's approximation) is equivalent to replace the wavelets of the spherical wavefronts by **parabolic wavefronts**.

The validity of this approximation is determined by the errors caused when the higher order terms of the development can no longer be neglected. A satisfactory condition would be that the maximum phase variation caused by  $a^2/8$  generates a phase variation much lower than 1 *radian*. The reader can verify that the condition for validity is the following [1 [[1]]]:

$$d_0 \gg \sqrt[3]{\frac{\pi}{4\lambda} \left[ (x - X_s)^2 + (y - Y_s)^2 \right]_{\max}^2}$$

In order to assign the order of magnitude, let us consider a source situated at  $(X_s, Y_s) = (0, 0)$  and a square observation region of 50mm on its side, the wavelength being equal to  $0.5 \mu\text{m}$ , we determine that  $d_0 \gg 1349 \text{ mm}$ .

## 2. Superposition of two electromagnetic waves: theoretical basics

### 2.1. Notations

Let us consider two electromagnetic waves represented by their electrical fields  $\vec{E}_1$  and  $\vec{E}_2$ . We note that  $\vec{k}_1$  and  $\vec{k}_2$  are their wave vectors,  $\omega_1$  and  $\omega_2$  their pulsations and  $\varphi_1$  and  $\varphi_2$  their original phases.

#### Exemple

Given a polarized field according to the direction  $\vec{e}_1$ , we have:

$$\vec{E}_1(\vec{r}, t) = A_1(\vec{r}) \exp\left[i(-\omega_1 t + \vec{k}_1 \cdot \vec{r} - \varphi_1)\right] \vec{e}_1$$

And given a polarized field according to the direction  $\vec{e}_2$ , we have:

$$\vec{E}_2(\vec{r}, t) = A_2(\vec{r}) \exp\left[i(-\omega_2 t + \vec{k}_2 \cdot \vec{r} - \varphi_2)\right] \vec{e}_2$$

For the field  $\vec{E}_2$ .

## 2.2. Resulting field from a superposition

Let us consider a zone in space where two waves co-exist. For all points in this space, the resulting electromagnetic field is simply equal to the sum of the fields:

$$\vec{E}(\vec{r}, t) = \vec{E}_1(\vec{r}, t) + \vec{E}_2(\vec{r}, t)$$

The observer, whether it be the naked eye or the detector, is sensitive to the square module of the electromagnetic field, i.e.  $|\vec{E}(\vec{r}, t)|^2$ . However, it must be taken into account that the field oscillates at a frequency  $\omega$  of the order of  $10^{14}$  Hz and that currently, even the best detectors cannot "see" phenomena that happen faster than  $10^{12}$  Hz. Consequently, the signal observed is a statistical average of  $|\vec{E}(\vec{r}, t)|^2$ , i.e. [2 [2]]

$$I(\vec{r}, t) = \langle |\vec{E}(\vec{r}, t)|^2 \rangle$$

where  $\langle \dots \rangle$  means statistical average.

## 2.3. The notion of coherence

The development of the previous expression yields:

$$I(\vec{r}, t) = \langle |\vec{E}_1(\vec{r}, t)|^2 \rangle + \langle |\vec{E}_2(\vec{r}, t)|^2 \rangle + 2 \Re e \left\{ \langle \vec{E}_1(\vec{r}, t) \cdot \vec{E}_2^*(\vec{r}, t) \rangle \right\}$$

We can note that:

$$\begin{cases} \langle |\vec{E}_1(\vec{r}, t)|^2 \rangle = \langle A_1^2(\vec{r}) \rangle = A_1^2(\vec{r}) \\ \langle |\vec{E}_2(\vec{r}, t)|^2 \rangle = \langle A_2^2(\vec{r}) \rangle = A_2^2(\vec{r}) \\ 2 \Re e \left\{ \langle \vec{E}_1(\vec{r}, t) \cdot \vec{E}_2^*(\vec{r}, t) \rangle \right\} = 2 A_1(\vec{r}) A_2(\vec{r}) \langle \cos((\omega_2 - \omega_1)t + (\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + \varphi_2 - \varphi_1) \rangle \vec{e}_1 \cdot \vec{e}_2 \end{cases}$$

The last term represents the term of interference of both fields, it is also called **interference modulation. This term is zero if:**

- the two waves are perpendicularly polarized, for example if  $\vec{e}_1 = \vec{e}_x$  and  $\vec{e}_2 = \vec{e}_y$  since then  $\vec{e}_1 \cdot \vec{e}_2 = 0$ ; we refer to this as **polarization coherence**
- both frequencies are very different  $\omega_1 \neq \omega_2$ , since the term in the cosine oscillates so quickly, at a frequency that approaches  $10^{14}$  Hz and it is not possible to observe with the naked eye or a detector; we refer to this as **temporal coherence**
- the wave vectors are highly variable in the region of observation  $\vec{k}_1(\vec{r}) \neq \text{constant}$  and  $\vec{k}_2(\vec{r}) \neq \text{constant}$ , since the term in the cosine oscillates very rapidly in space, leading to a null average, and it cannot be observed by the naked eye or by a detector; we refer to this as **spatial coherence**

These aspects of coherence will be further explained in a later part of the unit.

## 2.4. Interference signal

In the following example, let us consider two waves that are **temporally and spatially coherent, and also coherently polarized**. Given such a context, the observable signal is written simply as:

$$I(\vec{r}) = A_1^2(\vec{r}) + A_2^2(\vec{r}) + 2 A_1(\vec{r}) A_2(\vec{r}) \cos\left(\left(\vec{k}_1 - \vec{k}_2\right) \cdot \vec{r} + \varphi_2 - \varphi_1\right)$$

The spatial dependence with  $\vec{r}$  is implied subsequently whenever possible. If we assume  $I_1 = A_1^2$  and  $I_2 = A_2^2$ , the interference signal is also written as [3 [[3]]]:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\frac{2\pi\delta}{\lambda}\right)$$

with:

$$\frac{2\pi\delta}{\lambda} = \left(\vec{k}_1 - \vec{k}_2\right) \cdot \vec{r} + \varphi_2 - \varphi_1$$

where we take  $\delta$  as the optical path difference between the waves. The interference signal is also called fringe pattern or interference pattern.

The interference phenomenon is indicated by alternating dark zones for which  $\cos(\dots) = -1$  and bright zones for which  $\cos(\dots) = +1$ . Taking  $I_{\min}$  as the minimum signal and  $I_{\max}$  as the maximum signal, we have:

$$\begin{cases} I_{\min} = A_1^2 + A_2^2 - 2 A_1 A_2 \\ I_{\max} = A_1^2 + A_2^2 + 2 A_1 A_2 \end{cases}$$

We characterize the contrast of the fringe pattern by the visibility factor [3 [[3]]] :

$$C = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2 A_1 A_2}{A_1^2 + A_2^2}$$

The maximum contrast is equal to 1 and it is obtained for  $A_1 = A_2$ , which means that both waves have the same amplitude for all points in space. If  $A_1 \neq A_2$ , then the contrast is less than 1; and if  $A_1 \ll A_2$  or  $A_2 \ll A_1$ , then we obtain  $C \simeq 0$ .

The maxima for  $\cos(\dots) = +1$  occurs whenever:

$$\varphi = \left(\vec{k}_1 - \vec{k}_2\right) \cdot \vec{r} + \varphi_2 - \varphi_1 = 2k\pi \quad k \in \mathbb{R}$$

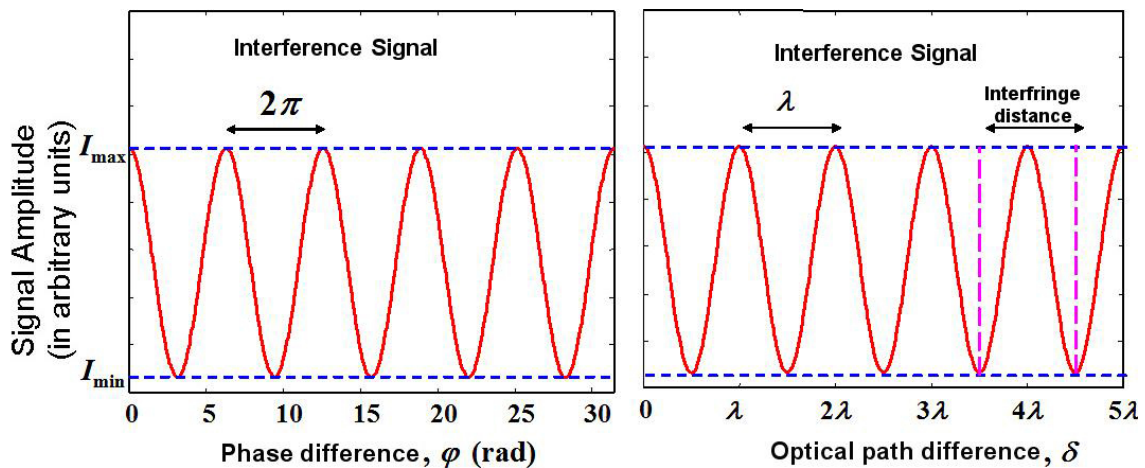
The minima obtained for  $\cos(\dots) = -1$ , occurs whenever

$$\varphi = \left(\vec{k}_1 - \vec{k}_2\right) \cdot \vec{r} + \varphi_2 - \varphi_1 = (2k+1)\pi \quad k \in \mathbb{R}$$

The  $\varphi$  is referred to as the phase of the interferences. It consists of the difference between the optical phases of the waves, and it is proportional to a difference in optical paths.

The distance, **at observation**, which separates **two consecutive zones of the same nature** is referred to as the **interfringe distance**.

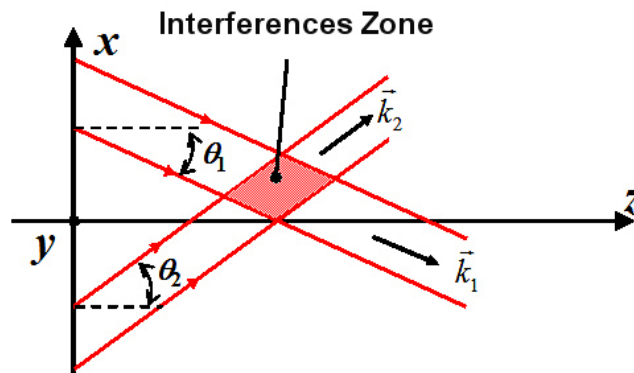
Figure 2 illustrates the properties of the interference signal.



We will now focus on two particular cases: when both waves are plane waves and when both waves are spherical waves.

## 2.5. Superposition of two plane waves

Let us consider two plane waves propagating towards  $z$  positives, and wave vectors that are not collinear to  $z$  but belong to the  $x, z$  plane. Figure 3 illustrates this scenario.



In this case, we simply have:

$$\vec{E}_1(\vec{r}, t) = A_1 \exp[i(-\omega t + \vec{k}_1 \cdot \vec{r} - \varphi_1)] \vec{e}_x$$

for the first wave and:

$$\vec{E}_2(\vec{r}, t) = A_2 \exp[i(-\omega t + \vec{k}_2 \cdot \vec{r} - \varphi_2)] \vec{e}_x$$

for the second one. Keeping in mind that:

$$\vec{k}_1 = \frac{2\pi}{\lambda} \cos \theta_1 \vec{e}_z - \frac{2\pi}{\lambda} \sin \theta_1 \vec{e}_x$$

$$\vec{k}_2 = \frac{2\pi}{\lambda} \cos \theta_2 \vec{e}_z + \frac{2\pi}{\lambda} \sin \theta_2 \vec{e}_x$$

and that :

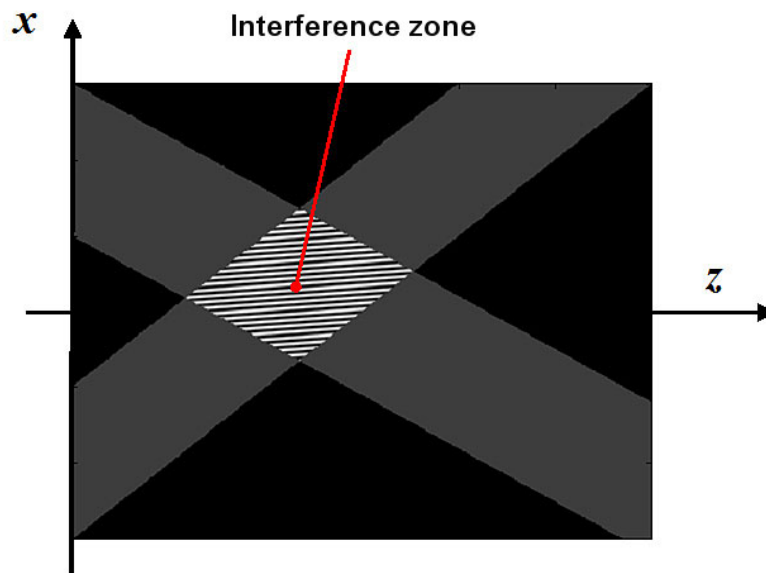
$$\varphi = (\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + \varphi_2 - \varphi_1 = \frac{2\pi}{\lambda} (\cos \theta_1 - \cos \theta_2) z - \frac{2\pi}{\lambda} (\sin \theta_1 + \sin \theta_2) x + \varphi_2 - \varphi_1$$

### Fondamental

the interference signal is written as:

$$I(x, y, z) = A_1^2 + A_2^2 + 2 A_1 A_2 \cos \left( \frac{2\pi}{\lambda} (\cos \theta_1 - \cos \theta_2) z - \frac{2\pi}{\lambda} (\sin \theta_1 + \sin \theta_2) x + \varphi_2 - \varphi_1 \right)$$

Figure 4 shows the interference field.



The distance that separates two consecutive zones of the same nature is the interfringe distance. For  $z$  constant, the abscissa  $x$  of the bright fringe at a given  $k$  order is such that:

$$\frac{2\pi}{\lambda} (\cos \theta_1 - \cos \theta_2) z_k - \frac{2\pi}{\lambda} (\sin \theta_1 + \sin \theta_2) x_k + \varphi_2 - \varphi_1 = 2k\pi$$

In the direction  $x$ , for the bright fringe that is consecutive to the preceding one, the optical phase varies from  $2\pi$  and we have:

$$\frac{2\pi}{\lambda} (\cos \theta_1 - \cos \theta_2) z_k - \frac{2\pi}{\lambda} (\sin \theta_1 + \sin \theta_2) x_{k+1} + \varphi_2 - \varphi_1 = 2k\pi + 2\pi$$

following the direction  $x$ , the distance that separates the two bright fringes (the distance between fringes in  $x$ ) is thus defined by:

$$\Delta x = \frac{\lambda}{\sin \theta_1 + \sin \theta_2}$$

where  $\Delta x = x_k - x_{k+1}$ .

Equivalently, in the direction  $z$ , it follows that the interfringe in  $z$  is:

$$\Delta z = \frac{\lambda}{\cos \theta_1 - \cos \theta_2}$$

where  $\Delta = z_{k+1} - z_k$ .

In the case that  $\theta_1 = \theta_2 = \theta$  it follows that:

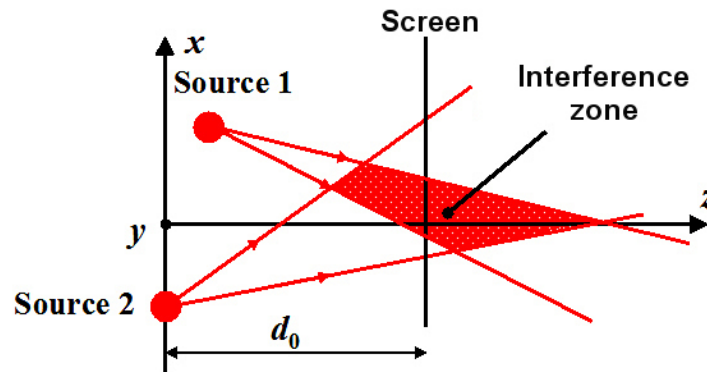
$$\begin{cases} \Delta x = \frac{\lambda}{2 \sin \theta} \\ \Delta z = \infty \end{cases}$$

Therefore, the fringes are parallel to the  $z$  axis.

To give some numerical values, let us consider  $\lambda = 0.5$  microns and  $\theta = 30^\circ$ , we obtain  $\Delta x = 0.5 / (2 \sin(30^\circ)) = 0.5$  microns. The interfringe is of the same order as the length of the waves. Therefore, moving from one bright fringe to another offers a sub-micrometric sensibility.

## 2.6. Superposition of two spherical waves

Let us consider two co-polarized (polarized along  $x$  direction) spherical waves propagating towards  $z$  positives. Figure 5 illustrates this scenario.



In this case, we simply have:

$$\vec{E}_1(\vec{r}_1, t) = \frac{A_1}{r_1} \exp(-i \omega t) \exp(i k r_1 - i \phi_1) \vec{e}_x$$

for the first wave, and:

$$\vec{E}_2(\vec{r}_2, t) = \frac{A_2}{r_2} \exp(-i\omega t) \exp(ikr_2 - i\varphi_2) \vec{e}_x$$

for the second one. Note that  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are the coordinates of the point sources. For all coordinates  $(x, y, z)$  we have:

$$r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2} \quad (i=1, 2)$$

The interference phase is  $\varphi = kr_1 - kr_2 + \varphi_2 - \varphi_1$ , i.e:

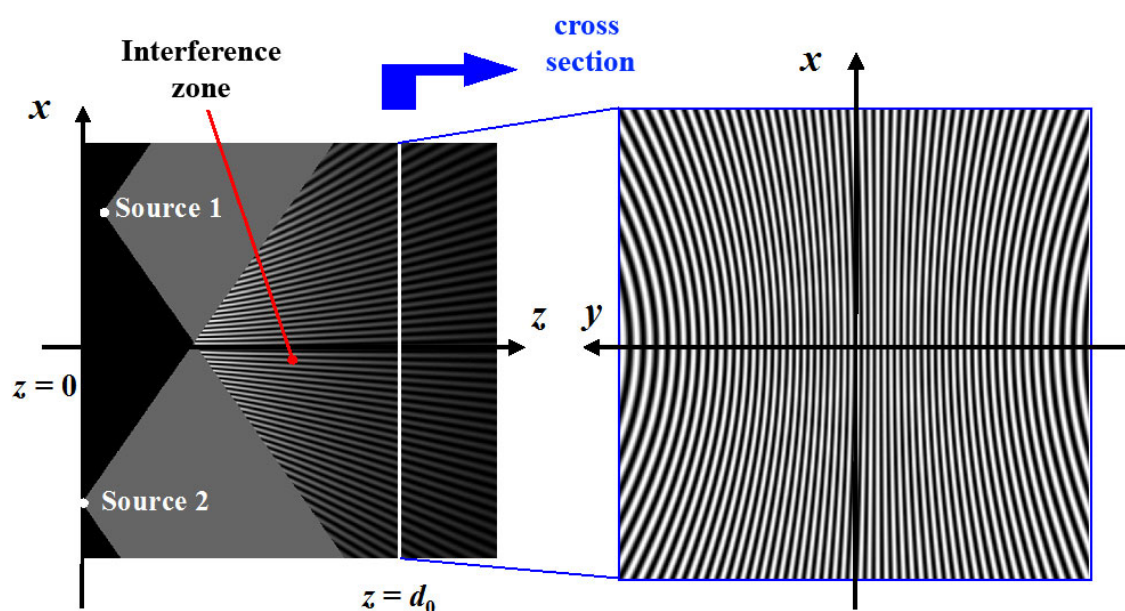
$$\varphi = \frac{2\pi}{\lambda} \left( \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} \right) + \varphi_2 - \varphi_1$$

## Fondamental

The interference signal is written as:

$$I(x, y, z) = \frac{A_1^2}{r_1^2} + \frac{A_2^2}{r_2^2} + 2 \frac{A_1 A_2}{r_1 r_2} \cos \left( \frac{2\pi}{\lambda} \left( \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} \right) + \varphi_2 - \varphi_1 \right)$$

Figure 6 shows the interference field in the superposition zone of the two waves on the  $(x, z)$  plane to the left, and on the cross-section to the right (plane  $(x, y, z = d_0)$ ).



We can see that the amplitude of interferences decreases with distance from the sources, due to the factor  $1/r_i$  of each term of the complex amplitude.

### Attention

Moreover, the fringes are curved, which was not the case for the interferences produced by plane waves. In the plane  $(x, y, z = d_0)$  the fringes are the arcs of a hyperbolae.

In space  $(x, y, z)$ , a bright fringe is localized by the condition:

$$\frac{2\pi}{\lambda} \left( \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} \right) + \varphi_2 - \varphi_1 = 2k\pi$$

i.e:

$$\left( \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} \right) = \alpha \lambda$$

with  $\alpha a = k + (\varphi_1 - \varphi_2)/2\pi$ . This relation is not simple to use.

However, we can write the interfringe distance explicitly in the case of a parabolic approximation. Suppose that  $d_1 = z - z_1$  and  $d_2 = z - z_2$ . Assuming this condition is true, a bright fringe is localized by the condition:

$$\frac{2\pi d_1}{\lambda} + \frac{\pi}{\lambda d_1} \left( (x-x_1)^2 + (y-y_1)^2 \right) - \frac{2\pi d_2}{\lambda} - \frac{\pi}{\lambda d_2} \left( (x-x_2)^2 + (y-y_2)^2 \right) + \varphi_2 - \varphi_1 = 2k\pi$$

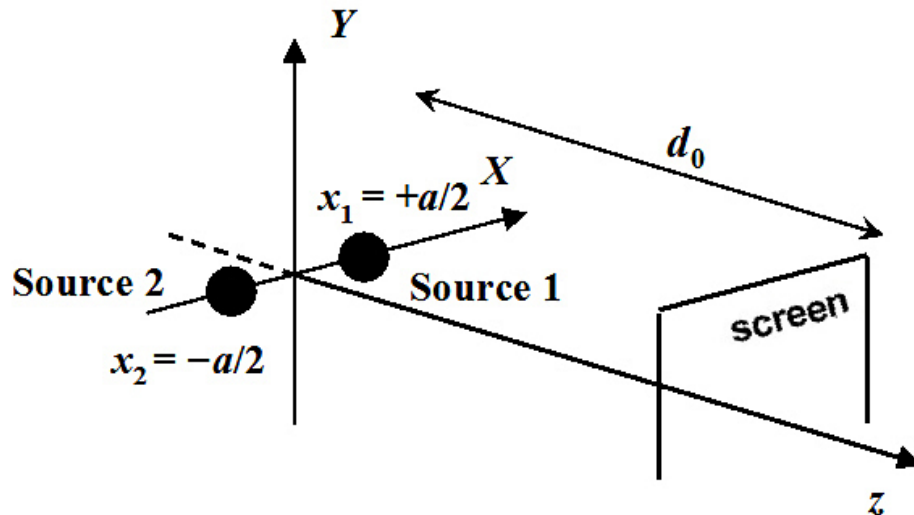
which gives:

$$\frac{x^2}{2} \left( \frac{1}{d_1} - \frac{1}{d_2} \right) + \frac{y^2}{2} \left( \frac{1}{d_1} - \frac{1}{d_2} \right) - x \left( \frac{x_1}{d_1} - \frac{x_2}{d_2} \right) - y \left( \frac{y_1}{d_1} - \frac{y_2}{d_2} \right) = \alpha' \lambda$$

with:

$$\alpha' = k + \frac{(\varphi_1 - \varphi_2)}{2\pi} - \frac{d_1 - d_2}{\lambda} - \frac{x_1^2 + y_1^2}{2\lambda d_1} + \frac{x_2^2 + y_2^2}{2\lambda d_2}$$

Let us consider an ideal case: suppose that the sources are localized symmetrically on the  $x$  axis, one at  $x_1 = +a/2$  and the other at  $x_2 = -a/2$  with  $y_1 = y_2 = 0$  and  $d_1 = d_2 = d_0$  and that the plane observation is defined as  $z = d_0 \gg a$ . Figure 7 illustrates this geometry.



In this case it simply follows that:

$$\frac{ax}{d_0} = \alpha' \lambda$$

With this configuration, the fringes are rectilinear and vertical, parallel to the  $y$  axis. We can see in figure 6 that this result corresponds to the center of the screen. The bright fringe of the order  $k$  is thus localized at  $x_k$  such that:

$$\frac{ax_k}{d_0} = \alpha' \lambda$$

And the next bright fringe of the order  $k + 1$  is localized at  $x_{k+1}$  such that:

$$\frac{ax_{k+1}}{d_0} = \alpha' \lambda + \lambda$$

The interfringe distance is thus:

$$i = x_{k+1} - x_k = \frac{\lambda d_0}{a}$$

With the parabolic approximation, the interfringe distance is a constant according to the direction  $x$ .

### *Remarque*

Figure 6 shows this result is true at the center of the screen, and that the interfringe distance increases relative to the distance from the center, because the fringes are arcs of hyperbolae.

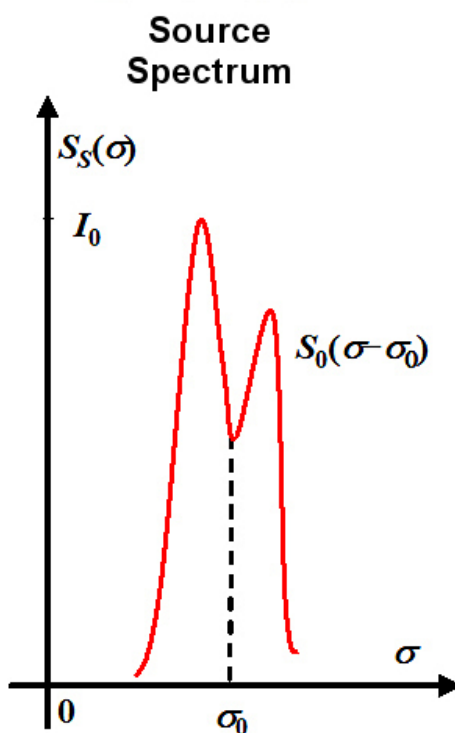
## 3. Temporal coherence

We saw in paragraph B.2 that the spectral content of the interfering waves influences the signal, especially the modulation term. In the case of two waves that are perfectly monochromatic and have the same wavelength, the fringes have the maximum contrast. This paragraph aims to study the influence of the wave spectral width on the fringe contrast. The influence of the **spectral width** describes the **temporal coherence** of light. The description of the source's spectral content is based on the notion of **spectral energy density**. This notion will be explained in the following.

### 3.1. Interferogram in polychromatic light

In the case of two waves that originate from a polychromatic source, the spectrum of the source no longer corresponds to a single ray located at wave number  $\sigma_0$  (or also at  $\lambda_0$  wave length,  $\nu_0$  frequency, or  $\omega_0$  angular frequency). The spectrum, or spectral energy density, denoted as  $S_s(\sigma)$ , is composed of an envelope  $S_0(\sigma)$  centered on the average wave number  $\sigma_0$ .

Figure 8 illustrates the spectral representation of the source.



According to paragraph B.4 "**Interferences signal**", the interference signal can be expressed as:

$$I(\delta, \sigma) = a S_s(\sigma) + b S_s(\sigma) \cos(2\pi \delta \sigma)$$

where:

$$a = I_1 + I_2 \quad b = 2\sqrt{I_1 I_2}$$

where  $\delta$  is the difference in the optical path difference between the waves, and from now on becomes:

$$I(\delta) = \int_0^{\infty} I(\sigma, \delta) d\sigma$$

The interferogram (or fringe pattern) is written as an integral [2 [[2]]]:

$$I(\delta) = a \int_0^{\infty} S_s(\sigma) d\sigma + b \int_0^{\infty} S_s(\sigma) \cos(2\pi \delta \sigma) d\sigma$$

The spectrum of the source, in other words the spectral representation of the source, is by nature real and zero for all values  $\sigma < 0$ . It follows therefore, that since  $S_s(\sigma)$  is real:

$$\int_0^{+\infty} S_s(\sigma) \cos(2\pi \delta \sigma) d\sigma = \Re \left[ \int_{-\infty}^{+\infty} S_s(\sigma) \exp(-2i\pi \delta \sigma) d\sigma \right] = \Re \left[ \tilde{S}_s(\delta) \right]$$

where  $\tilde{S}_s(\delta)$  is the Fourier transform of the spectral density, and we write:

$$I(\delta) = a \Re \left[ \tilde{S}_s(0) \right] + b \Re \left[ \tilde{S}_s(\delta) \right]$$

### Attention

**This result, therefore, means that the interferogram observed in the presence of the polychromatic light is directly related to the Fourier transform of the source's spectrum.**

Qualitatively, if the source has a narrow spectrum, then the interferogram will be large and spread out; and if the source has a very large spectrum, then the interferogram will be localized at  $\delta = 0$ . We express the spectrum in the following manner:

$$S_s(\sigma) = S_0(\sigma - \sigma_0) = S_0(\sigma) * \delta(\sigma - \sigma_0)$$

where  $\delta(\sigma)$  is the Dirac distribution. We thus have:

$$\Re \left[ \tilde{S}_s(\delta) \right] = \Re \left[ \tilde{S}_0(\delta) \times \exp(-2i\pi \sigma_0 \delta) \right] = \Re \left[ \tilde{S}_0(\delta) \right] \cos(2\pi \sigma_0 \delta)$$

At  $\delta = 0$ , we have  $\Re \left[ \tilde{S}_s(0) \right] = \Re \left[ \tilde{S}_0(0) \right]$  and the distribution of intensity as a function of optical path difference  $\delta$  becomes:

$$I(\delta) = \Re \left[ \tilde{S}_0(0) \right] \left( a + b \Re \left[ \frac{\tilde{S}_0(\delta)}{\tilde{S}_0(0)} \right] \cos(2\pi \sigma_0 \delta) \right)$$

The cosine's term of modulation is referred to as the degree of temporal coherence. We note the degree of temporal coherence:

$$\gamma_{c,t}(\delta) = \Re \left[ \frac{\tilde{S}_0(\delta)}{\tilde{S}_0(0)} \right]$$

## Fundamental

This result constitutes the **Wiener–Kinchine Theorem** [2 [[2]]]: it demonstrates that the degree of temporal coherence at point  $M$  on the screen is equal to the numerical value of the Fourier transform of the spectral energy density, calculated to the value of the optical path difference of point  $M$ . The degree of temporal coherence is normalized in such a way that  $\gamma_{c,t} = 1$  if the source is monochromatic; if the source only emits a single wavelength, the temporal coherence is obviously perfect.

### 3.2. Coherence length

The coherence length of the source is defined by the relationship to the degree of coherence. We have:

$$l_c = \int_{-\infty}^{+\infty} |\gamma_{c,t}(\delta)|^2 d\delta$$

The source's coherence time is thus given by:

$$\tau_c = \frac{l_c}{c}$$

where  $c$  is the velocity of the light in vacuum. The coherence time can be interpreted as the average temporal width of a wave front emitted by the source.

#### Remarque

At an observation point  $M$ , if the optical path difference is greater than the coherence length, then the interference phenomenon can no longer be observed. The illumination appears uniformly white. This shade is called higher order white. When this illumination is analyzed, we can observe a **discrete spectrum**, which means that the illumination is missing some wavelengths; these correspond to destructive interferences.

### 3.3. Examples

Let us consider several sources and their corresponding interferograms [2 [[2]],4 [[4]]] :

#### a) Case of monochromatic radiation

In the case where the spectrum is only composed of a monochromatic wave with a wavelength  $\lambda_0$ , we can model the spectrum as:

$$S_s(\sigma) = I_0 \delta(\sigma - \sigma_0)$$

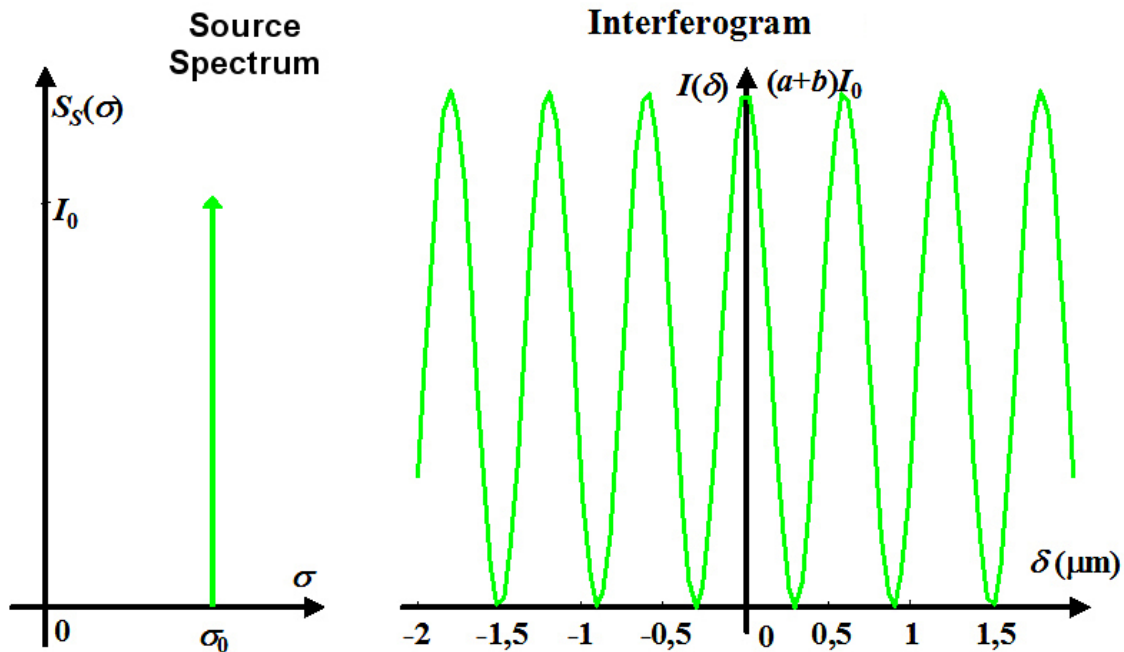
Therefore, the spectral envelope is reduced to a Dirac distribution  $S_0(\sigma) = I_0 \delta(\sigma)$ . The Fourier transform  $S_0(\sigma)$  is  $\widetilde{S}_0(\delta) = I_0$ . The degree of coherence is then:

$$\gamma_{c,t}(\delta) = 1$$

The coherence length is thus infinite, since the degree of coherence cannot be integrated. The interference signal is then:

$$I(\delta) = a I_0 + b I_0 \cos(2\pi \sigma_0 \delta)$$

The spectrum and its interferogram are represented in figure 9 for a monochromatic source with a wavelength of  $\lambda_0 = 600 \text{ nm}$ .



### b) Case of two wavelengths

If the source includes two monochromatic waves with wave numbers  $\sigma_1$  and  $\sigma_2$  with  $\sigma_2 > \sigma_1$  and equal amplitude  $I_0$ , its spectrum can be modeled as:

$$S_S(\sigma) = I_0 \delta(\sigma - \sigma_1) + I_0 \delta(\sigma - \sigma_2)$$

In this case, the average wave number of the spectrum is  $\sigma_0 = \frac{1}{2}(\sigma_1 + \sigma_2)$  and the spectral interval is  $\Delta\sigma = \sigma_2 - \sigma_1$ . We then have:

$$S_S(\sigma) = I_0 \delta\left(\sigma - \sigma_0 - \frac{\Delta\sigma}{2}\right) + I_0 \delta\left(\sigma - \sigma_0 + \frac{\Delta\sigma}{2}\right)$$

and since  $S_0(\sigma)$  is the spectral envelope (see figure 8), we have:

$$S_0(\sigma) = I_0 \delta\left(\sigma - \frac{\Delta\sigma}{2}\right) + I_0 \delta\left(\sigma + \frac{\Delta\sigma}{2}\right)$$

for which the Fourier transform is:

$$\tilde{S}_0(\delta) = 2I_0 \cos(\pi \Delta\sigma \delta)$$

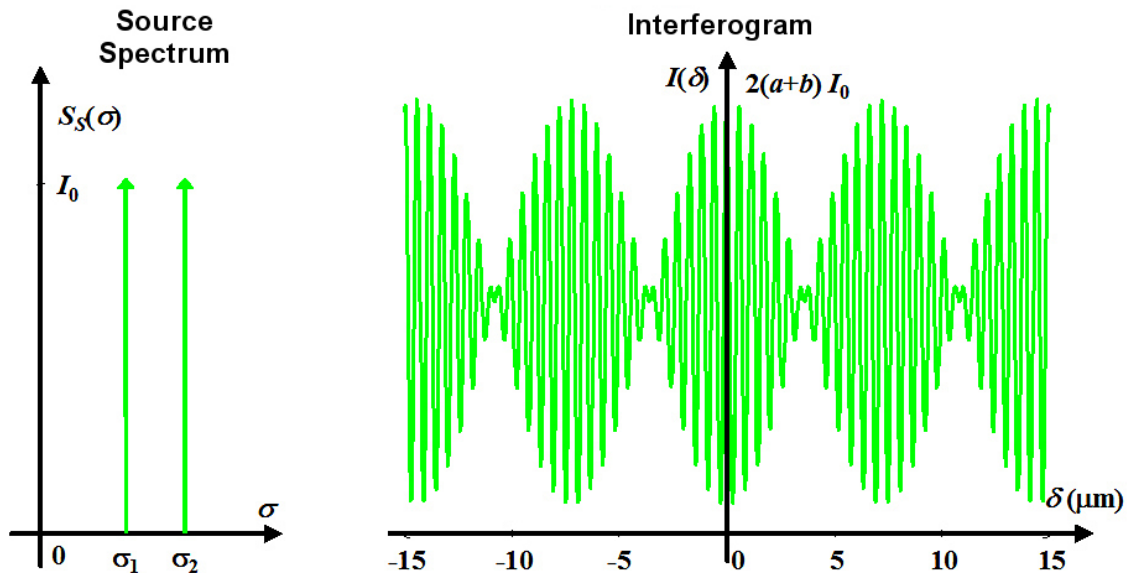
The degree of temporal coherence is thus:

$$\gamma_{c,t}(\delta) = \cos(\pi \Delta\sigma \delta)$$

The interference signal is written as:

$$I(\delta) = 2a I_0 + 2b I_0 \cos(\pi \Delta \sigma \delta) \cos(2\pi \sigma_0 \delta)$$

The interferogram shows **beats**. They are represented in figure 10 in the case that  $\lambda_1 = 600 \text{ nm}$  and  $\lambda_2 = 650 \text{ nm}$ .



In this ideal case where the line width is considered as infinitely narrow, the coherence length is also infinite because the degree cannot be calculated. However, in reality, it is necessary to consider the width of each line as well as a model for the lineshape. We would then see that the degree of coherence can be calculated, and that the coherence length is not infinite but depends significantly on the line width.

### c) Case of a rectangular spectrum distribution

If we consider a uniform spectrum distribution on the interval  $\Delta\sigma$  centered on  $\sigma_0$  and with amplitude  $I_0$ , the spectrum is described by the rectangular function:

$$S_s(\sigma) = I_0 \Pi_{\Delta\sigma}(\sigma - \sigma_0)$$

and:

$$S_0(\sigma) = I_0 \Pi_{\Delta\sigma}(\sigma)$$

for which the Fourier transform is:

$$\tilde{S}_0(\delta) = \Delta\sigma I_0 \text{sinc}(\Delta\sigma \delta)$$

### Remarque

for the following equations, the *sinc* function is defined by

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

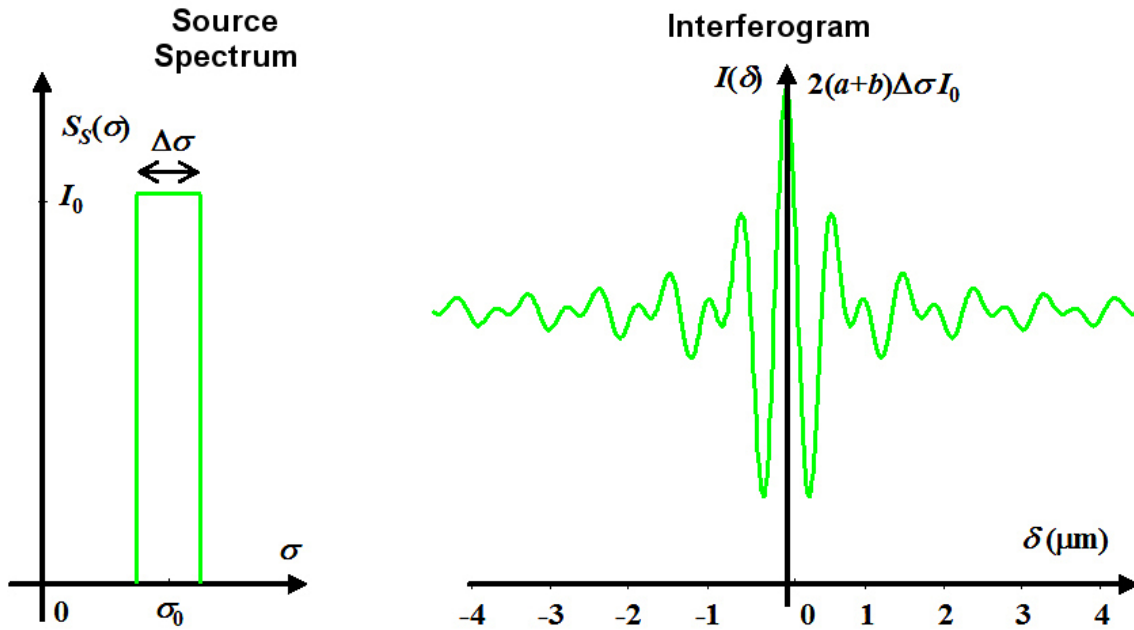
The intensity distribution is written as:

$$I(\delta) = I_0 \Delta \sigma (a + b \operatorname{sinc}(\Delta \sigma \delta) \cos(2\pi \sigma_0 \delta))$$

The degree of coherence of a given source is thus:

$$\gamma_{c,t}(\delta) = \operatorname{sinc}(\Delta \sigma \delta)$$

The spectrum and the interferogram are represented in figure 11, in the case where  $\lambda_0 = 600 \text{ nm}$  and  $\Delta \lambda_0 = 400 \text{ nm}$  (simplified model of white light).



The coherence length is obtained by calculating:

$$l_c = \int_{-\infty}^{+\infty} |\operatorname{sinc}(\Delta \sigma \delta)|^2 d\delta$$

which gives:

$$l_c = \frac{1}{\Delta \sigma} = \frac{\lambda^2}{\Delta \lambda}$$

With the numerical values, we find that  $l_c = 0.9 \text{ microns}$  and  $\tau_c = 0.3 \text{ fs}$ .

#### d) Case of a monochromatic wave with a lorentzian spectral distribution

This situation corresponds to the broadening of spectral lines by the Doppler effect, which is caused by thermal agitation in the emitting medium. The spectral intensity can be modeled by:

$$S_s(\sigma) = \frac{I_0}{1 + 4(\sigma - \sigma_0)^2 / \Delta \sigma^2}$$

where  $\Delta \sigma$  represents the length of the line at mid-height.

The envelope of the spectrum allocation is described by:

$$S_0(\sigma) = \frac{I_0}{1 + 4\sigma^2/\Delta\sigma^2}$$

for which the Fourier transform is:

$$\tilde{S}_0(\delta) = \frac{\pi}{2} \Delta\sigma I_0 \exp(-\pi \Delta\sigma |\delta|)$$

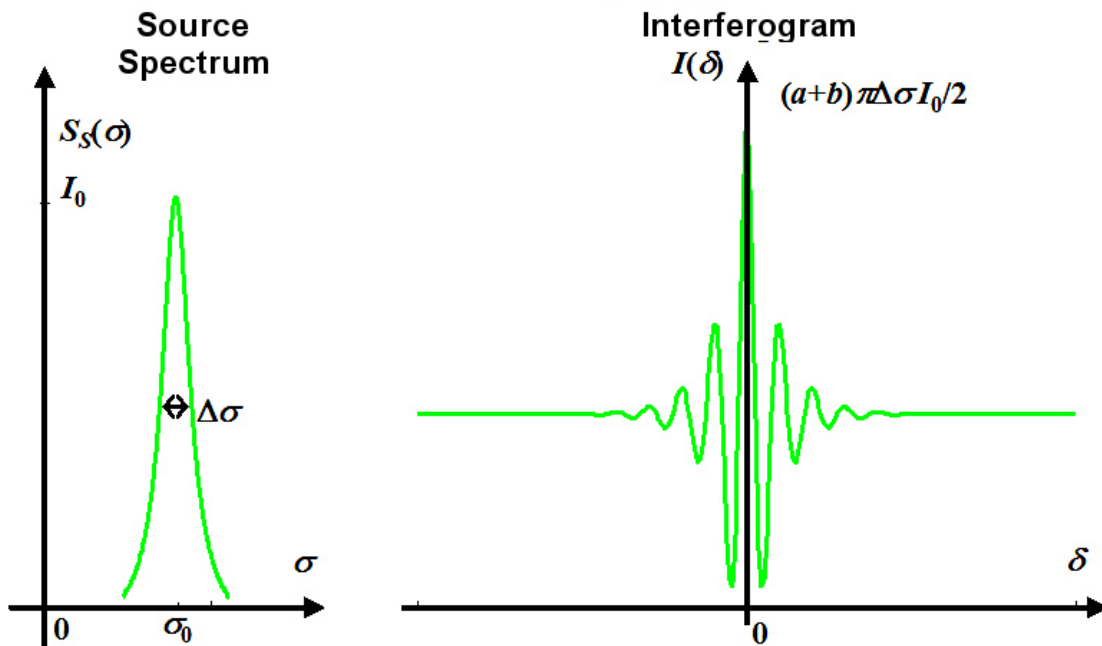
The degree of temporal coherence is:

$$\gamma_{c,t}(\delta) = \exp(-\pi \Delta\sigma |\delta|)$$

The intensity is expressed as:

$$I(\delta) = \frac{\pi}{2} \Delta\sigma I_0 (a + b \exp(-\pi \Delta\sigma |\delta|) \cos(2\pi \sigma_0 \delta))$$

The wider the line is and the faster the interferogram decreases, then the ability to observe the signal modulation also decreases. The spectrum and the interferogram are represented schematically in figure 12.



The coherence length is:

$$l_c = \int_{-\infty}^{+\infty} |\exp(-\pi \Delta\sigma |\delta|)|^2 d\delta$$

which gives:

$$l_c = \frac{1}{\pi \Delta\sigma} = 0.318 \frac{\lambda^2}{\Delta\lambda}$$

For example, in the case of a diode laser for which  $\lambda_0 = 669 \text{ nm}$  and  $\Delta\nu = 1000 \text{ MHz}$  we have  $\Delta\lambda = 1.2 \times 10^{-4} \text{ nm}$ . We can calculate that  $l_c = 95.4 \text{ cm}$  and  $\tau_c = 3.18 \text{ ns}$ . With the parameters, the graphic representation of the interferogram of an optical path difference equal to the coherence length is not possible to the large number of oscillations of  $\cos(2\pi\sigma_0\delta)$  under the modulation curve.

## 4. Spatial coherence

We saw in paragraph B.2 that interfering wave vectors influence the signal and in particular the cosine term. In the case of two waves that are perfectly monochromatic and of the same wave length, the interference signal is written as:

$$I = a + b \cos(2\pi \delta / \lambda) = a + b \cos(2\pi \delta \sigma)$$

This paragraph aims to study the influence of the spatial width of the source, where the waves are issued from, on the contrast of the fringe figures. The influence of the **spatial width** of the source describes the **spatial coherence** of light. All interferometric devices seek to generate two waves from a single wave that is produced by a single source. There are many diverse technical solutions to do this, but we can always identify two secondary sources from which the waves that will produce the interferences propagate. In the following section, we will consider the presence of two secondary sources without going into the technical means that are necessary for their creation. The course, **Interference: interferometers**, will discuss interferometers more in depth. If the reader wishes to know more about this subject, he may consult the aforementioned course.

### 4.1. Interferogram with an extended source

In the case of two waves originating from the same extended source, a wave vector corresponds to each point in the source. Thus, each point in the source will contribute to form the interferogram. Therefore, we can see in the observation zone the superposition of all the interferograms that are produced by the all the points in the source.

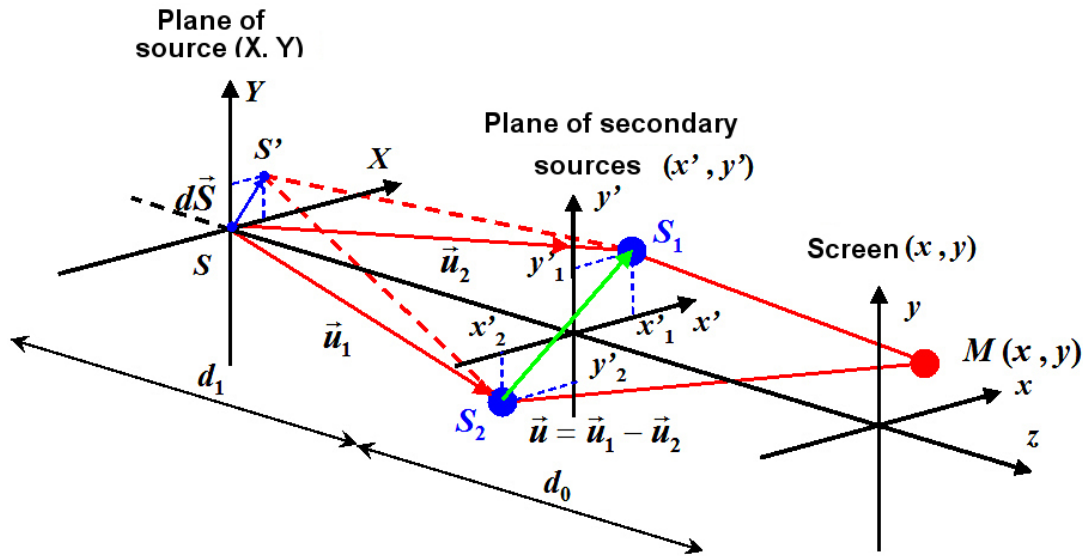
#### Remarque

Qualitatively, if all the interferograms are slightly dephased, one by one, the interference signal will be noisy, which will mean a drop in fringe modulation. This drop in modulation is of the same kind as that of temporal coherence: the amplitude of the cosine is weighed by the degree of spatial coherence of the source.

All points in the source are characterized by their luminance. Each surface element of the source,  $dXdY$ , centered on  $S$ , produces, after the interferometer, a light distribution of the type [2 [[2]]]:

$$dI \propto L(S) dX dY \times (a + b \cos(2\pi \delta \sigma)) = L(S) dX dY \times (a + b \Re\{\exp(2i\pi \delta \sigma)\})$$

The virtual "synchronous" sources that were created from a single source by the interferometer are referred to as "secondary sources." In figure 13 the two secondary sources are represented by  $S_1$  and  $S_2$ . The secondary sources  $S_1$  and  $S_2$  have their respective coordinates  $(x'_1, y'_1)$  and  $(x'_2, y'_2)$ . The width of the source is materialized by the vector  $\vec{dS} = \vec{SS'}$ .



\* Figure 13: The shift of the source point of  $d\vec{S}$  produces a variation in the optical path difference.

## 4.2. Evaluation of the optical path difference

For any point  $S'$  in the source, the optical path difference at point  $M$  at observation coordinates  $(x, y)$  can be broken down into two parts. The first one takes into account the optical path difference introduced between the source and the secondary sources, while the second one takes into account the optical path difference between the secondary sources and observation point  $M$ .

$$\delta(S') = ([S'S_2] - [S'S_1]) + ([S_2M] - [S_1M])$$

with:

$$[S'S_2] - [S'S_1] = \Delta(X, Y) = \sqrt{(X - x'_2)^2 + (Y - y'_2)^2 + d_1^2} - \sqrt{(X - x'_1)^2 + (Y - y'_1)^2 + d_1^2}$$

Under the condition that the transversal lengths remain small despite the distance separating the primary source and the two secondary sources  $(|d\vec{S}|, |S_1S_2| \ll d_1)$ , we can write:

$$\Delta(X, Y) = d_1 + \frac{1}{2d_1} \left( (X - x'_2)^2 + (Y - y'_2)^2 \right) - d_1 - \frac{1}{2d_1} \left( (X - x'_1)^2 + (Y - y'_1)^2 \right)$$

which gives:

$$\Delta(X, Y) = \frac{x'_2{}^2 + y'_2{}^2}{2d_1} - \frac{x'_1{}^2 + y'_1{}^2}{2d_1} + \frac{X(x'_1 - x'_2) + Y(y'_1 - y'_2)}{d_1}$$

We can therefore deduce that:

$$\delta(S') = \delta_0 + \frac{X(x'_1 - x'_2) + Y(y'_1 - y'_2)}{d_1}$$

where

$$\delta_0 = ([S_2 M]) - ([S_1 M]) + \frac{x_2'^2 + y_2'^2}{2d_1} - \frac{x_1'^2 + y_1'^2}{2d_1}$$

### Remarque

For point S, and the secondary sources placed symmetrically, we find that:

$$\delta(S) = ([S_2 M]) - ([S_1 M])$$

Based on the previous expressions, we can note that optical path difference can also be written as:

$$\delta(S') = \delta_0 + \frac{1}{d_1} (\vec{u}_1 - \vec{u}_2) \cdot d\vec{S}$$

where  $\vec{u}_1 = \overrightarrow{SS_1}$  and  $\vec{u}_1 - \vec{u}_2 = \overrightarrow{S_2S_1}$ .

This expression leads us to the following conclusions:

1. There is no variation between the optical path difference and the width of the source when:
  - The shift  $d\vec{S}$  is perpendicular to the vector  $\vec{u} = \vec{u}_1 - \vec{u}_2$ . In this case, the source point can be replaced by a source slit without affecting the fringe contrast.
  - If the two vectors  $\vec{u}_1$  and  $\vec{u}_2$  are collinear, the interferometer is said to be amplitude-splitting, and a wide source can be used.
2. The optical path difference in  $S'$  is expressed as:  $\delta(S') = \delta_0 + \alpha X + \beta Y$   
 where  $(\alpha, \beta) = ((x'_1 - x'_2)/d_1, (y'_1 - y'_2)/d_1)$  are the apparent angles through which the two secondary sources are seen from the source's center S in both transversal directions of the reference axes.
3. When the vector  $\vec{u}$  is parallel to the shifting vector  $d\vec{S}$ , the optical path difference is expressed as:  $\delta(S') = \delta_0 + \alpha \Delta S$

Fringe contrast decrease will occur, if the variation in optical path difference is equal to

$$\lambda/2, \text{ i.e.: } \alpha \Delta S \leq \frac{\lambda}{2}$$

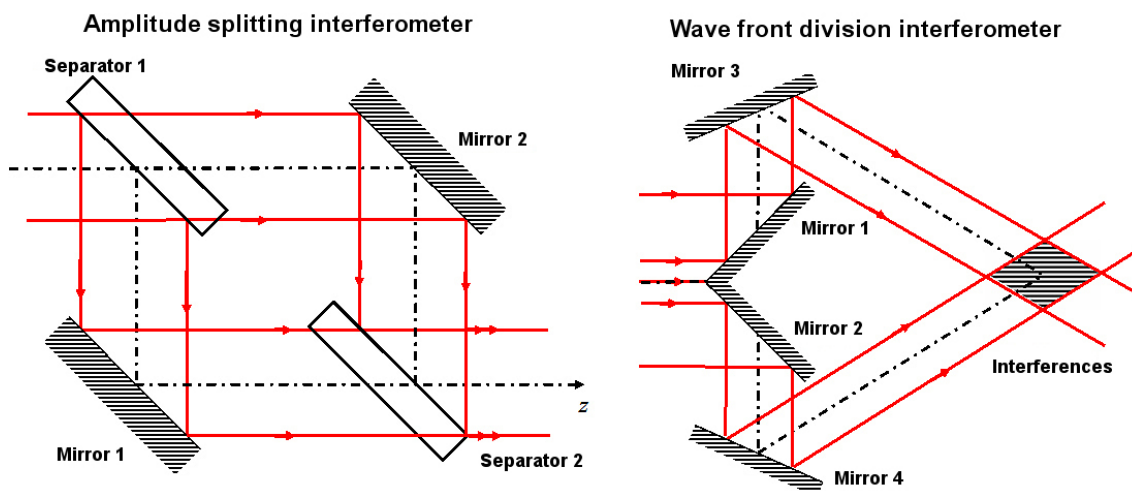
which corresponds to an angular width of spatial coherence of the order:  $\alpha_c \leq \frac{\lambda}{2\Delta S}$ .

### 4.3. Two types of interferometers

In the previous paragraph, we saw that there are amplitude-splitting interferometers that are characterized by the fact that the unit vectors  $\vec{u}_1$  and  $\vec{u}_2$  are collinear; this means that the interfering rays correspond to two emissions from the interferometer produced by the same incident ray.

The other variety of interferometers are wavefront-splitting interferometers. There are inconvenient because they cannot be used with a wide source, since the fringes rapidly

become confused with the dimension of the source. For both kinds of interferometers, the interferences are not localized, that is to say they are not observable in all of the field of view, when the source is non extended. On the other hand, in the case of a wide source with an amplitude-splitting interferometer the interferences are localized at the intersections of pairs of emerging rays corresponding to the same incident ray.



The table beneath summarizes the previous sections.

	Wavefront-splitting	Amplitude-splitting
Puntual source	Non-localized interferences	Non-localized interferences
Extended source	No interferences	Localized interferences

In the module “**Interference: interferometers**” the reader may find different examples and analyses of such devices.

### 4.4. Illumination at an observation point

The source element  $dXdY$  gives to observation point  $M$  of the abscissa  $(x, y)$  the intensity:

$$dI \propto L(S') dX dY \times (a + b \Re \{ \exp(-2i\pi\delta(S')\sigma) \})$$

i.e. by replacing  $\delta(S')$  with the expression:

$$dI \propto L(X, Y) dX dY \times (a + b \Re \{ \exp(-2i\pi\sigma(\delta_0 + \alpha X + \beta Y)) \})$$

For all of the source, the different points emit incoherently:

$$I = \int_{source} dI$$

i.e.:

$$I = a \int_{source} L(X, Y) dX dY + b \Re \{ \exp(2i\pi\sigma\delta_0) \int_{source} L(X, Y) \exp(-2i\pi\sigma(\alpha X + \beta Y)) dX dY \}$$

This expression can be rewritten as:

$$I = a \int_{-\infty}^{+\infty} L(X, Y) dX dY + b \cos(2\pi\sigma\delta_0) \Re \left\{ \int_{-\infty}^{+\infty} L(X, Y) \exp(2i\pi(\sigma\alpha X + \sigma\beta Y)) dX dY \right\}$$

we note that:

$$I = a \Re \left[ \tilde{L}(0,0) \right] + b \Re \left[ \tilde{L}(\sigma\alpha, \sigma\beta) \right] \cos(2\pi\sigma\delta_0)$$

where  $\tilde{L}(\sigma\alpha, \sigma\beta)$  is the Fourier transform of the source's light distribution. This result means that the interferogram observed in the presence of an extended source is directly related to the Fourier transform of the source's light distribution.

For  $\alpha = \beta = 0$ , we have  $\tilde{L}(0) = \Re[\tilde{L}(0)]$  and the interferogram becomes:

$$I = \Re \left[ \tilde{L}(0,0) \right] \left( a + b \Re \left[ \frac{\tilde{L}(\sigma\alpha, \sigma\beta)}{\tilde{L}(0,0)} \right] \cos(2\pi\sigma\delta_0) \right)$$

The modulation term of the cosine is referred to as the degree of spatial coherence of the source. We note the spatial degree of coherence:

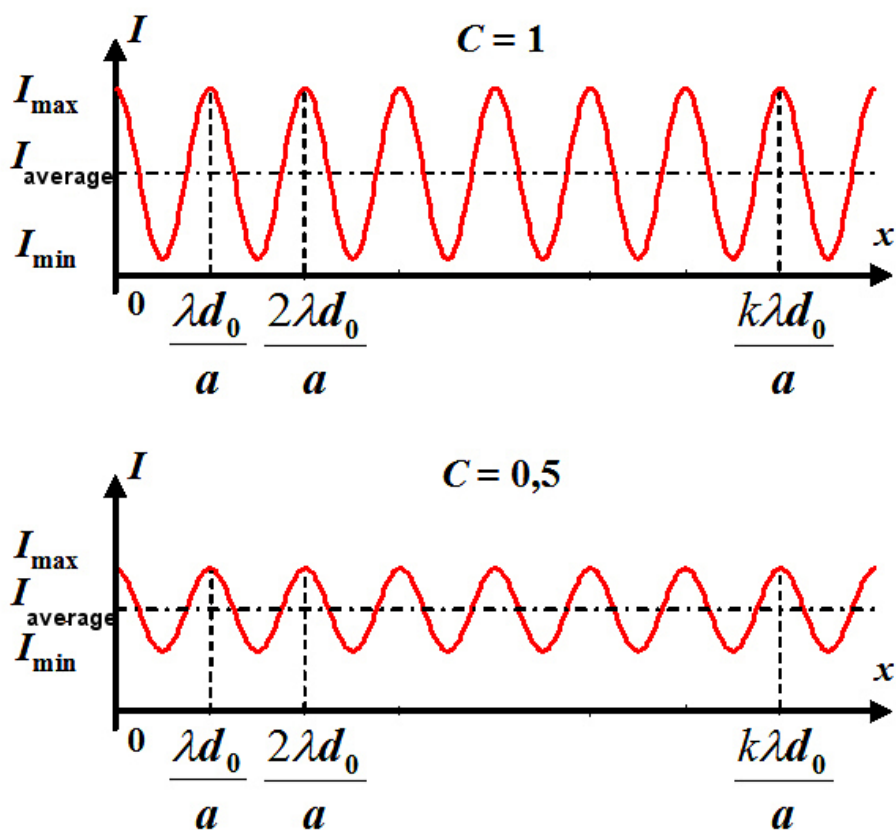
$$\gamma_{c,s}(\alpha, \beta) = \Re \left[ \frac{\tilde{L}(\sigma\alpha, \sigma\beta)}{\tilde{L}(0,0)} \right]$$

## Fundamental

This result constitutes the **Van Cittert–Zernike Theorem** [2 [[2]]] : it demonstrates that the degree of spatial coherence at point M on the screen is equal to the value of the Fourier transform of the source's light distribution that is calculated for the angular coordinates where the secondary sources are seen from the center of the primary source. The degree of spatial coherence is normalized in such a way that  $\gamma_{c,s} = 1$  if  $S_1$  is confused with  $S_2$  the spatial coherence is obviously perfect if the secondary sources are superimposed.

## Remarque

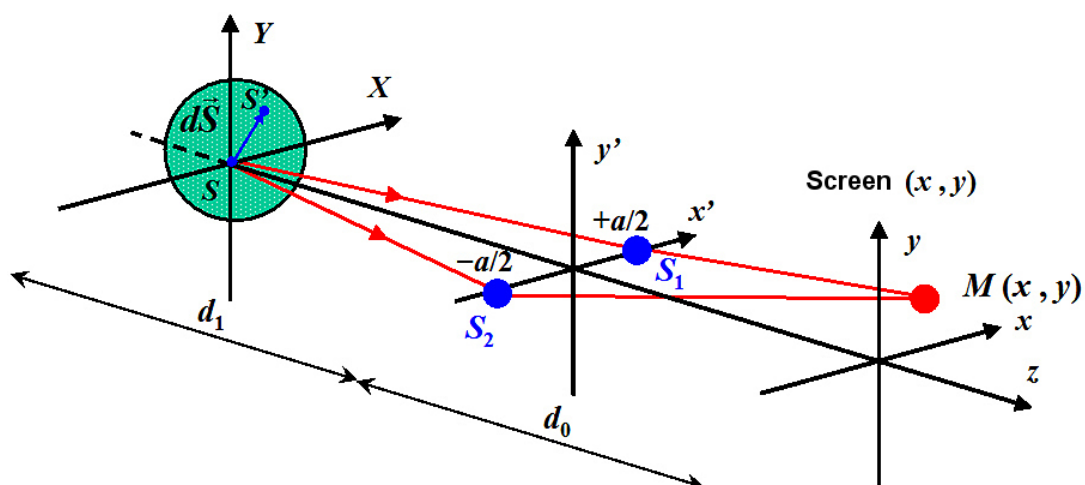
Note that unlike temporal coherence, **the contrast does not depend on the value of the optical path difference**, but that instead it depends on the angle from which the secondary sources are seen from the center of the source. The contrast diminishes as angles  $\alpha$  and  $\beta$  increase. (figure 15).



## 4.5. Examples

### a) The case of a circular source

Let us examine the decrease in contrast produced by an extended source with a circular shape that is emitting a monochromatic radiation of  $\lambda = 500 \text{ nm}$ . Figure 16 shows the notations. We note  $2R$  the diameter of the source.



The circular source has a uniform brightness that can be described by the following function:

$$L(X, Y) = \begin{cases} 1 & X^2 + Y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of the source's angular distribution is given by:

$$\tilde{L}(\sigma \alpha, \sigma \beta) = \int_{-\infty}^{+\infty} L(X, Y) \exp(-2i\pi(\sigma \alpha X + \sigma \beta Y)) dX dY$$

The source has a circular symmetry; we can use a transformation in polar coordinates for  $(X, Y)$  and  $(\alpha, \beta)$ , i.e.:

$$\begin{cases} r = \sqrt{X^2 + Y^2} & x = r \cos \theta & y = r \sin \theta & \theta = \arctan(Y/X) & \theta \in [0, 2\pi] \\ \rho = \sqrt{\alpha^2 + \beta^2} & \alpha = \rho \cos \phi & \beta = \rho \sin \phi & \phi = \arctan(\beta/\alpha) & \phi \in [0, 2\pi] \end{cases}$$

thus,

$$\tilde{L}(\sigma \alpha, \sigma \beta) = \tilde{L}(\rho, \phi) = \int_0^{2\pi} d\theta \int_0^{+\infty} r L(r) \exp[-2i\pi\sigma r \rho (\cos \theta \cos \phi + \sin \theta \sin \phi)] dr$$

or also:

$$\tilde{L}(\rho, \phi) = \int_0^{+\infty} r L(r) \left[ \int_0^{2\pi} \exp[-2i\pi\sigma r \rho \cos(\theta - \phi)] d\theta \right] dr$$

considering that:

$$J_0(2\pi r \rho) = \frac{1}{2\pi} \int_0^{2\pi} \exp[-2i\pi\sigma r \rho \cos(\theta - \phi)] d\theta$$

$J_0$  being the Bessel function of the first kind of order 0, we have:

$$\tilde{L}(\rho, \phi) = 2\pi \int_0^{+\infty} r L(r) J_0(2\pi\sigma r \rho) dr$$

Due to the definition of the brightness function, it follows that:

$$\tilde{L}(\rho, \phi) = 2\pi \int_0^{+R} r J_0(2\pi\sigma r \rho) dr$$

Assuming  $u = 2\pi\sigma r \rho$ , we obtain:

$$\tilde{L}(\rho, \phi) = \frac{1}{2\pi\sigma^2\rho^2} \int_0^{+2\pi\sigma\rho R} u J_0(u) du$$

along with the equality:

$$\int_0^x u J_0(u) du = x J_1(x)$$

$J_1$  being the Bessel function of order 1 such that  $J_1(0) = 0$ , we finally obtain:

$$\tilde{L}(\rho, \phi) = R \frac{J_1(2\pi\sigma\rho R)}{\sigma\rho} = \pi R^2 \frac{2J_1(2\pi\sigma\rho R)}{2\pi\sigma\rho R}$$

Since we have (figure 16):

$$\rho = \sqrt{\alpha^2 + \beta^2} = \sqrt{(x'_1 - x'_2)^2/d_1^2 + (y'_1 - y'_2)^2/d_1^2} = a/d_1$$

the degree of spatial coherence of the circular source is finally written as ( $2J_1(0)/0 = 1$ ):

$$\gamma_{c,s} = \Re \left[ \frac{\tilde{L}(\sigma\alpha, \sigma\beta)}{\tilde{L}(0,0)} \right] = \Re \left[ \frac{\tilde{L}(\rho, \phi)}{\tilde{L}(0,0)} \right] = \frac{2J_1\left(\frac{2\pi a R}{\lambda d_1}\right)}{\frac{2\pi a R}{\lambda d_1}}$$

The formula for the degree of spatial coherence of the circular source is  $2J_1(Z)/Z$  where  $Z = 2\pi a R/\lambda d_1$ . Figure 17 shows the outline of the degree of spatial coherence of the circular source as a function of the value of  $Z$  expressed in *radians*.

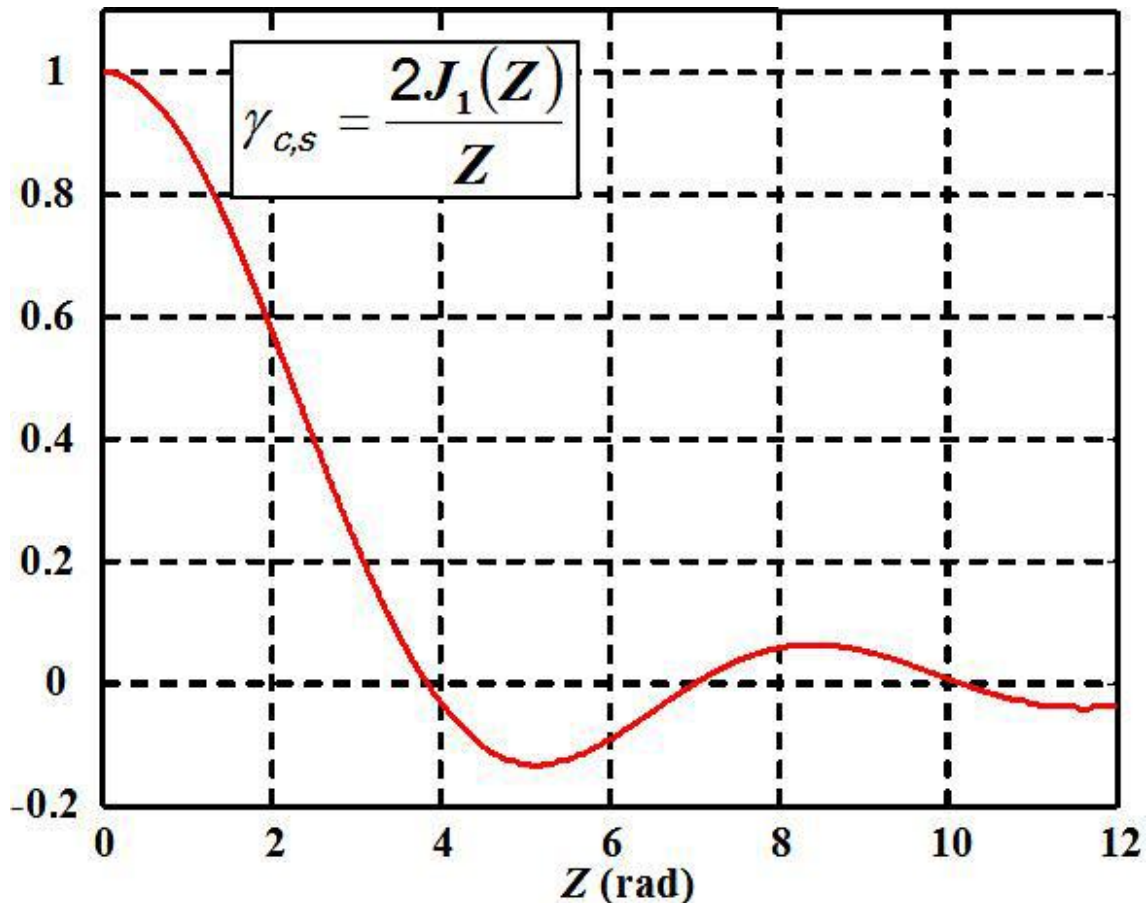


Table 2 gives the values of  $Z$  for maxima, minima, and zeros of the degree of coherence.

$Z$ (rad)	$\frac{2J_1(Z)}{Z}$	max / min / zero
0	1	max
3.832	0	zero
5.136	-0.1323	min
7.015	0	zero
8.416	0.0648	max
10.172	0	zero
11.620	-0.04	min

The zeroes of the degree of spatial coherence are not equidistant. The first contrast cancellation is obtained by:

$$\frac{2\pi a R}{\lambda d_1} = 3.832$$

i.e. by a distance between the secondary sources that is equal to:

$$a = 1.22 \frac{\lambda d_1}{2R}$$

Remember that  $2R$  represents the source diameter.

As a result of the angular diameter  $\alpha_s = 2R/d_1$  from which the source is seen from the plane of the secondary sources, we obtain:

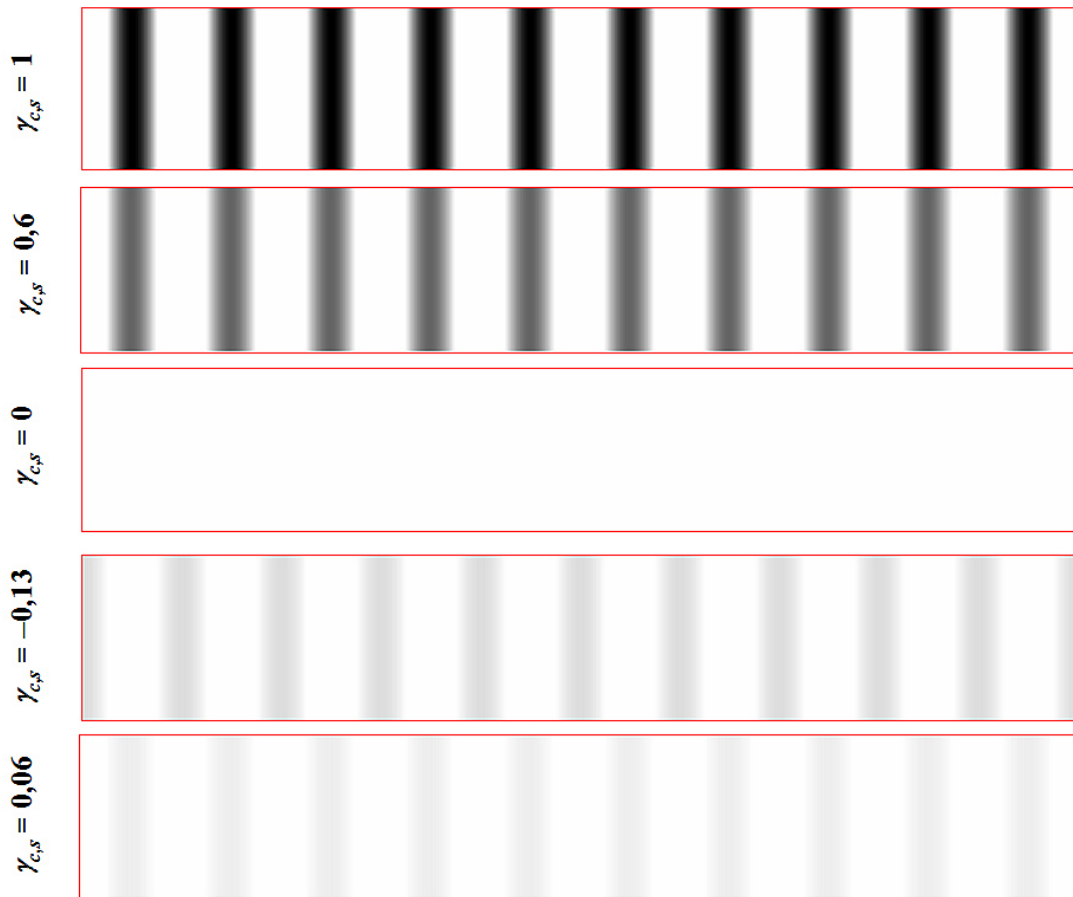
$$a = 1.22 \frac{\lambda}{\alpha_s}$$

As a numerical example, let us consider the case of the Sun for which the apparent angle of observation from Earth is  $\alpha_{\text{Sun}} = 32' = 9 \times 10^{-3}$  rad. The preceding relation gives  $a \approx 68$  microns for  $\lambda = 0.5$  microns. If we want to observe contrasted interferences at a point **M** on the screen, the secondary sources must not be separated by a distance greater than this value. Sunlight, therefore, has a weak spatial coherence.

For secondary sources of the same amplitude, the interference signal at a point **M** on the screen can be written as:

$$I(M) = I_0 \left( 1 + \frac{2J_1(2\pi a R / \lambda d_1)}{2\pi a R / \lambda d_1} \cos(2\pi \sigma \delta_0) \right)$$

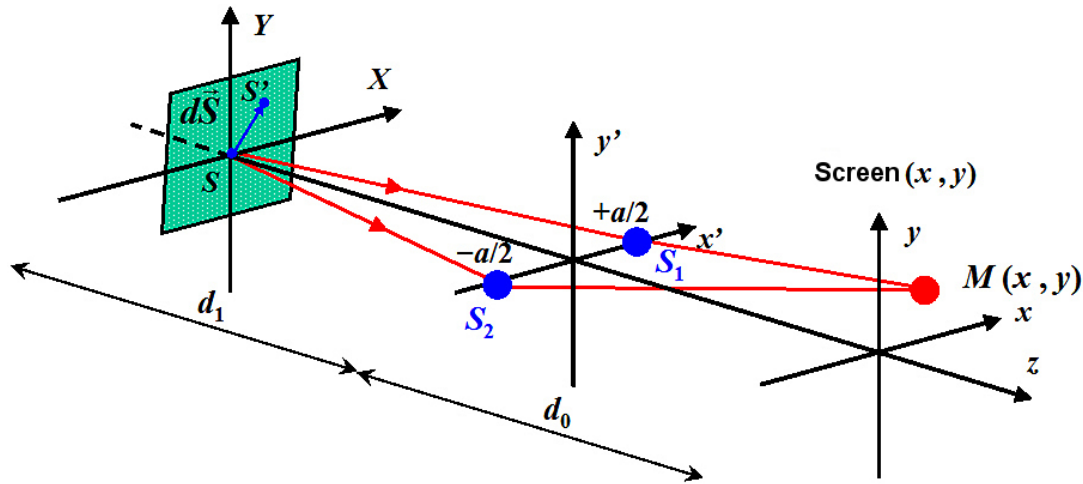
Figure 18 shows the interferograms obtained for the different values of the degree of coherence on a screen.



The fringe contrast evolves as a result of the source radius  $R$  following the curve in figure 17. If the value of  $R$  is low, the fringe contrast is ostensibly equal to the unit. As  $R$  increases, the contrast diminishes and eventually becomes null when  $a = 1.22\lambda d_1/2R$ . Then, when the contrast becomes negative, the bright fringes occupy the position of the dark fringes and vice versa; this inversion can be observed quite easily in figure 18, even if the contrast is rather weak ( $|\gamma_{c,s}| = 0.13$ ). If the value of  $R$  continues to increase, the fringes become less and less visible until they disappear completely.

### b) The case of a rectangular source

Now let us examine the decrease in contrast caused by an extended source with a rectangular shape. Figure 19 shows the notations. We note  $(L_x, L_y)$  as the dimensions of the source.



The circular source has a uniform luminance that can be described by the following function:

$$L(X, Y) = \begin{cases} 1 & |X| \leq L_x/2 \text{ and } |Y| \leq L_y/2 \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of the light distribution of the source is given by:

$$\tilde{L}(\sigma \alpha, \sigma \beta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} L(X, Y) \exp(-2i\pi(\sigma \alpha X + \sigma \beta Y)) dX dY$$

Keeping in mind the definition of the brightness function, we have:

$$\tilde{L}(\sigma \alpha, \sigma \beta) = \int_{-L_x/2}^{+L_x/2} \exp(-2i\pi\sigma \alpha X) dX \int_{-L_y/2}^{+L_y/2} \exp(-2i\pi\sigma \beta Y) dY$$

This expression can be easily integrated, so we find that:

$$\tilde{L}(\sigma \alpha, \sigma \beta) = L_x L_y \text{sinc}(\sigma L_x \alpha) \text{sinc}(\sigma L_y \beta)$$

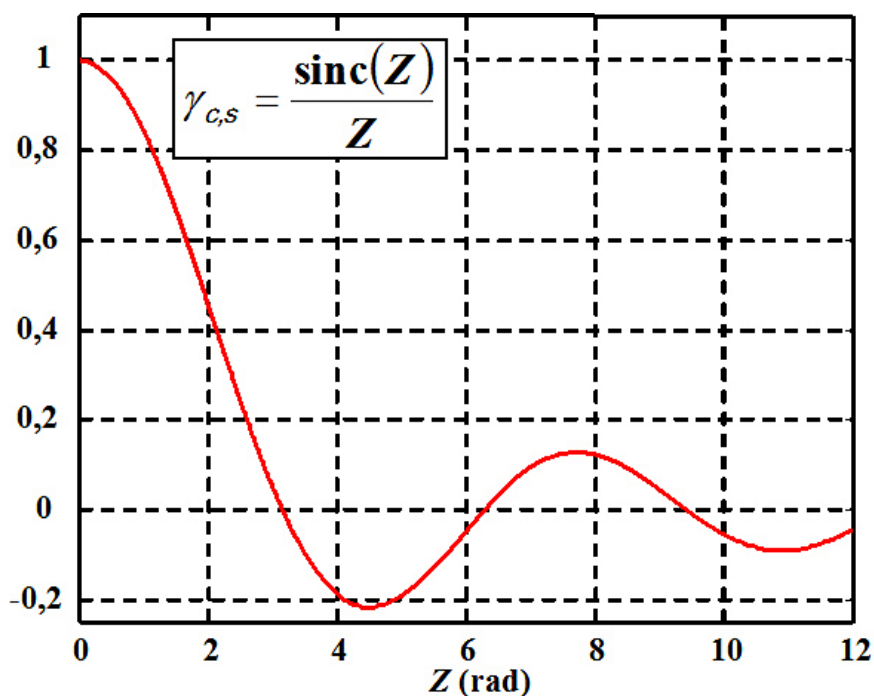
Since the secondary sources are reduced to two points,  $(\alpha, \beta) = (a/d_1, 0)$ , we have:

$$\tilde{L}(\sigma \alpha, 0) = L_x L_y \text{sinc}\left(\frac{a L_x}{\lambda d_1}\right)$$

For the case illustrated in figure 19, the degree of spatial coherence of the rectangular source is written as:

$$\gamma_{c,s} = \Re \left[ \frac{\tilde{L}(\sigma \alpha, \sigma \beta)}{\tilde{L}(0,0)} \right] = \text{sinc}\left(\frac{a L_x}{\lambda d_1}\right)$$

The degree of spatial coherence of a rectangular source has the shape of a *sinc* function. Figure 20 shows the shape of the degree of spatial coherence of the rectangular source as a function of the value of  $Z$  expressed in radians.



The zeroes of the degree of coherence are equidistant. The contrast is cancelled at:

$$\frac{\pi a L_x}{\lambda d_1} = n\pi \quad n \in \mathbb{N}^*$$

The first zero is obtained for the distance between the secondary sources that is equal to:

$$a = \frac{\lambda d_1}{L_x}$$

This expression gives us an estimate of the width of the spatial coherence of the source.

\* \*  
\*

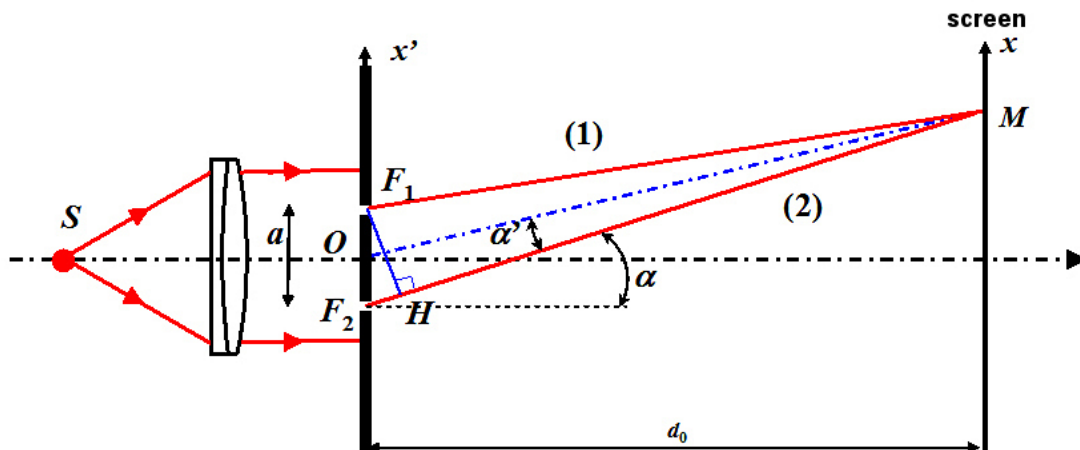
This course has introduced two-wave interferences, in order to understand these it also examined the notions of optical path difference and coherence. The reader should remember the following key points: temporal coherence is due to the spectral width, while spatial coherence is related to the source's extension; a monochromatic and point source is coherent both temporally and spatially.

The case studies will focus on Young's slits interferometer used with monochromatic light, and with white light.

The exercises will test your understanding of the main ideas presented in the course.

## II. Case studies : interferences with perfectly coherent light

The examples and case studies of two-wave interferences can be further explained through Young's Double Slit experiment for a temporally coherent and spatially coherent source. The general results that are found remain valid for any experimental model chosen. We will use these results again in the module "**Interference: interferometers.**" Young's experiment is represented schematically in figure 21.



A point source  $S$ , which is coherent spatially, is placed at the focal point of a lens; the resulting plane wave intercepts an opaque sheet with two identical and very small slits  $F_1$  and  $F_2$  (for example 0.01 mm in diameter), separated by distance  $a$ . The wave length of the monochromatic plane wave is noted as  $\lambda$ . Due to diffraction, a phenomenon that is not studied in this course, each slit  $F_1$  and  $F_2$  behaves like a source lighting a screen that is placed further on in front of them. The radiation emitted by  $F_1$  and  $F_2$  recombine at a point  $M$  on the screen, after having gone through two different optical paths (1) and (2). A point of the screen with slits is identified by its coordinates  $(x', y')$ . The observation is then made on screen  $E$  situated at distance  $d_0$  from the Young slits. This distance is large compared to distance  $a$ ; a point  $M$  on the screen is identified by its coordinates  $(x, y)$ . Since the two waves are temporally coherent, the intensity distribution on the screen is given by the relation (see "**Interferences signal**", paragraph B.4)

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\frac{2\pi \delta}{\lambda}\right)$$

For both source slits that are diffracting the light in the same way, we have  $I_1 = I_2 = I_0/2$  where  $I_0$  is the intensity of the source and the interference signal is written as:

$$I = \frac{I_0}{2} \left(1 + \cos\left(\frac{2\pi \delta}{\lambda}\right)\right) \quad (2)$$

The optical path difference, denoted as  $\delta$ , is due to the fact that the optical path  $[F_2M]$  is different from the optical path  $[F_1M]$ . This optical path difference produces the phase shift  $\varphi$  given by:

$$\varphi = \frac{2\pi\delta}{\lambda}$$

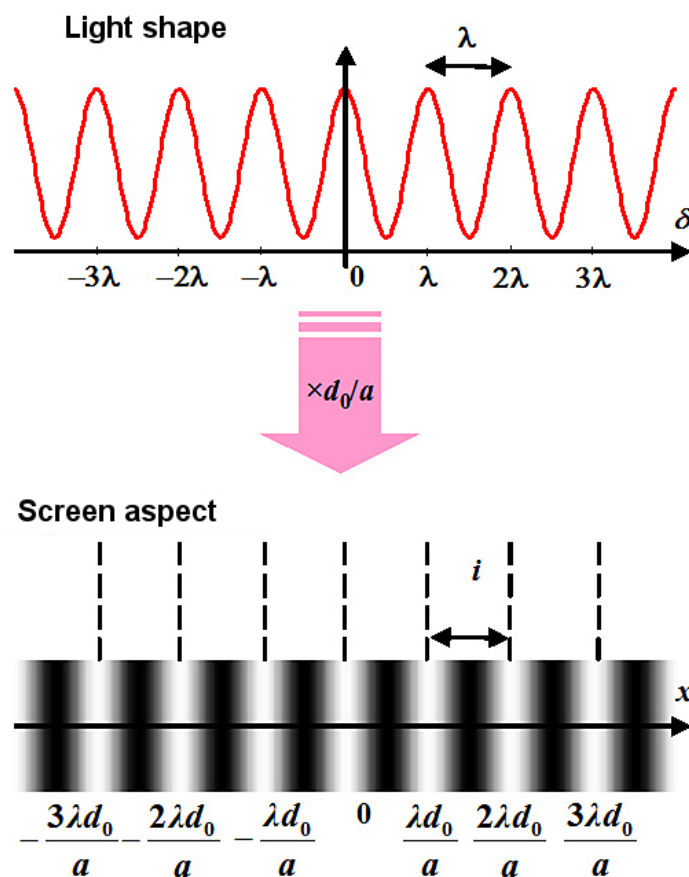
This general relation can be applied in all cases where we have:

- a point source (spatial coherence),
- a monochromatic source (temporal coherence),
- two interfering waves.

The distinction between interference experiments for two waves only affects the expression of the optical path difference  $\delta$  which only depends on the geometric parameters of the set-up.

The intensity distribution  $\frac{I}{I_0}(\delta)$  is with an average value of  $1/2$  and with period  $\lambda$ . It is represented in figure 22. For the bright points on the screen, the optical path difference  $\delta$  is equal to a whole number multiplied by the wavelength:  $\delta = k\lambda (k \in \mathbb{N})$ . For the dark points, the optical path difference  $\delta$  is equal to an odd whole number multiplied by half the wavelength:  $\delta = (2k + 1)\lambda/2 (k \in \mathbb{N})$ .

The quotient of  $k = \delta/\lambda$  is referred to as the **order of interference**. All the points of the same order of interference are in the same state of interference; they belong to the same fringe, and therefore have the same illumination.



## 1. Young's slit experiment with monochromatic light

In this paragraph we will examine how to express  $\delta$  as a function of the abscissa  $x$  of observation point  $M$ , of the distance  $a$  of the slits, and of the distance  $d_0$  from the slits to the observation screen. In this set-up, the distance  $d_0$  in meters is far larger than the distance between the axes  $a$  of the slits in millimeters. It is possible to consider that the distance  $F_1M$  is equal to the distance  $MH$ , which is the same as identifying the cord  $F_1H$  with the corresponding arc. Since the index of air is very close to 1, the optical path difference  $\delta$  is equal to length  $F_2H$  :

$$\delta = F_2H = a \sin \alpha$$

The distance  $d_0$  is also large for  $x$ , the angles  $\alpha$  and  $\alpha'$  are small and slightly different from one another, so if  $\alpha$  is expressed in radians:

$$\tan \alpha \approx \tan \alpha' = \frac{x}{d_0} \approx \sin \alpha \approx \alpha$$

i.e. by combining the last two equations:

$$\delta = \frac{a x}{d_0}$$

and replacing  $\delta$  with its expression in relation (2), we obtain an expression for the interference signal:

$$I = \frac{I_0}{2} \left( 1 + \cos \left( \frac{2 \pi a x}{\lambda d_0} \right) \right)$$

The spatial period on the screen, referred to as **interfringe distance** is expressed as:

$$i = \frac{\lambda d_0}{a}$$

It represents the distance on the screen between two consecutive bright fringes or two consecutive dark fringes. If the distance  $a$  between the slits increases, the interfringe distance  $i$  diminishes; the interfringe distance also diminishes with the wavelength.

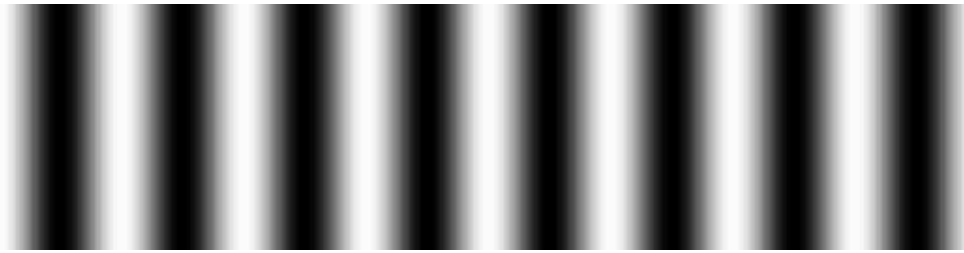
You can see an animation of these influences at the following website: <http://www.f-vandenbrouck.org/interference1.html><sup>1</sup>

Since  $a$ ,  $d_0$ ,  $\lambda$  are constants for any given set-up, the site of the points that are in the same state of interference and have thus the same order of interference, verifies that:

$$\frac{\delta}{\lambda} = k = C^{te} \rightarrow x = C^{te}$$

Therefore, site of these points is constituted by a set of planes parallel to the constant abscissa. On the screen, these planes present rectilinear fringes that are parallel to each other (see figure 23).

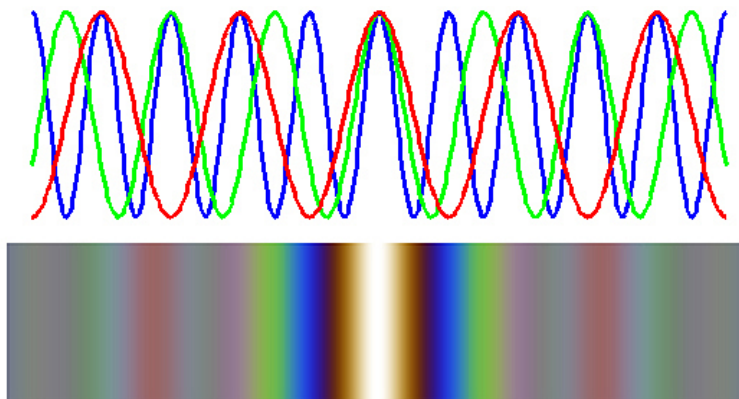
1 - <http://www.f-vandenbrouck.org/interference1.html>



## 2. Case of white light

For white light, we can consider that each wavelength of the visible spectrum constitutes a system of superimposed fringes. As long as the optical path difference  $\delta$  remains smaller than the **coherence length** of the source, the field appears as colored fringes (see figure 24) referred to as white light fringes. When  $\delta$  increases, the colors fade and we refer to them as bleached. Then, when the optical path difference becomes greater than the coherence length, the field appears uniformly white. This shade is referred to as **higher order white**. The fringes have disappeared, in other words, the modulation in the cosine has been reduced by the effect of the source's temporal coherence. For a white source where  $\lambda_0 = 600 \text{ nm}$  and  $\Delta\lambda = 400 \text{ nm}$  (see paragraph C.4 - **Case of a rectangular spectrum distribution**), the

coherence length is given by  $l_c = \frac{\lambda^2}{\Delta\lambda} = 0,9 \mu\text{m}$ , thus when the optical path difference exceeds 1 micron, the fringes disappear.



### Remarque

If we analyze high order white color with a spectrum, we will see that all wavelengths between 400 nm and 800 nm are absent; those that verify the relation  $\delta = (2k + 1)\lambda/2 (k \in \mathbb{N})$  and that are hence "turned off" by destructive interference are missing.

# III. Self-Check exercises

## 1. Course content

Indicate which statements are true:

Question 1

[Solution n°1 p 43]

The longer the lifespan of a wave front, the more monochromatic the light.

Question 2

[Solution n°2 p 43]

The longer the lifespan of a wave front, the weaker the frequency.

Question 3

[Solution n°3 p 43]

The longer the lifespan of a wave front, the larger the coherence length.

Question 4

[Solution n°4 p 43]

In emptiness, the triad  $(\vec{k}, \vec{E}, \vec{B})$  is orthogonal for an electromagnetic wave.

Question 5

[Solution n°5 p 43]

A parallel beam of light represents a plane wave.

Question 6

[Solution n°6 p 43]

Wave surfaces contain light rays.

Question 7

[Solution n°7 p 43]

Quasi monochromatic light has a large spectrum.

Question 8

[Solution n°8 p 43]

The coherence length of a quasi monochromatic laser (gas) source  $\Delta\lambda \simeq 10^{-6}$  nm is large, up to several meters.

Question 9

[Solution n°9 p 43]

A wave surface is the site of all points where the electrical field vectors oscillate at the same phase (modulo  $2\pi$ )

Question 10

[Solution n°10 p 43]

Two wave surfaces at a distance of  $2\lambda$  oscillates at the same phase (modulo  $4\pi$ )

Question 11

[Solution n°11 p 44]

Two polarized rectilinear waves whose polarization directions intersect at an angle of  $90^\circ$  can interfere.

## Question 12

[Solution n°12 p 44]

Two polarized rectilinear wave whose polarization directions intersect at an angle of  $45^\circ$  can interfere.

## Question 13

[Solution n°13 p 44]

The fringe contrast of interferences is the greatest when the two sources have the same intensity.

## 2. Young experiment

In the Young slit experiment, the slits are spaced by distance of  $a = 0.2 \text{ mm}$  and the screen is placed at a distance of  $d_0 = 1.5 \text{ m}$ . The distance between the fifth minima situated on both sides of the fringe of the order of 0 is equal to  $34.73 \text{ mm}$ .

## Question

[Solution n°14 p 44]

Determine the wavelength of the used source.

## 3. Young experiment with white light

A source of white light ( $400 - 700 \text{ nm}$ ) illuminates Young slits spaced  $1.25 \text{ mm}$  away. An interference pattern can be observed on a screen that is placed  $1.5 \text{ m}$  away. If the source slit of a high-resolution spectroscopy is placed at the level of the screen and  $3 \text{ mm}$  away from the central white fringe (and parallel to the interference fringes), we can observe dark grooves.

## Question 1

[Solution n°15 p 44]

At point **M** can we observe interferences ? Explain.

## Question 2

[Solution n°16 p 45]

To what wavelengths do the grooves correspond?

## 4. Problem

Let us consider two slits  $F_1$  and  $F_2$  made on an opaque screen that are infinitely narrow, vertical, and spaced by  $a = 2 \text{ mm}$ . They are illuminated by a point source  $S$ , monochromatic with a wavelength of  $\lambda_0 = 550 \text{ nm}$ . The source slit  $S$  is equidistant to  $F_1$  and  $F_2$ . The interference pattern is observed on a screen  $E$ , which is parallel to the slits and  $d_0 = 1 \text{ m}$  away from them.

## Question 1

[Solution n°17 p 45]

Calculate the illumination on screen **E**. Determine the interfringe distance and the contrast.

## Question 2

[Solution n°18 p 46]

How does the preceding result change if  $S$  is replaced by a vertical, infinitely narrow slit  $F_0$  that is parallel to  $F_1$  and  $F_2$  ?

## Question 3

[Solution n°19 p 46]

What can be observed if we introduce a constant phase shift  $\varphi_0$  on one of the trajectories? At  $F_1$  we interpose a very thin glass plate with parallel faces, of index  $n = 1.5$ . When this plate is placed on screen  $E$ , all of the fringes translate by  $t = 2.9$  mm.

1. What is the direction of this translation?
2. How many fringes are affected by this?
3. Deduce the thickness  $e$  of the glass plate.

## Question 4

[Solution n°20 p 47]

We replace the plate with an absorbing screen with an optical density of 2 ( $DO = 2$ ). Determine the new contrast.

## Question 5

[Solution n°21 p 47]

We remove the density filter and now use a spatially incoherent source. Describe the phenomenon that is observed if the source is polychromatic (white light).

## Question 6

[Solution n°22 p 47]

The source is  $d_1 = 100$  mm away from  $F_1$  and  $F_2$  and has a width of  $e = 6.8$   $\mu\text{m}$ . What happens to the light at point  $M$  on screen  $E$ ? How does the visibility vary with  $e$ ? What is the maximum width for which the decrease in contrast remains below 10%?

## Question 7

[Solution n°23 p 48]

We use as source a grid made up of slits that are parallel to  $F_1$  et  $F_2$ . Determine the grid spacing and the width of the slits for which the contrast remains of the preceding value.

## Question 8

[Solution n°24 p 49]

The source is once again a slit reduced to  $F_0$  but the source of monochromatic light is replaced by a sodium vapor source, which has a spectrum essentially made up of two radiations with a wavelength of  $\lambda_1 = 589.3$  nm and  $\lambda_2 = 589.9$  nm.

Find the energy spectral density function  $S_s(\sigma)$  in terms of the average wave number  $\sigma = (\sigma_1 + \sigma_2)/2$  and the deviation of the doublet  $\Delta\sigma = \sigma_1 - \sigma_2$ .

Find the illumination normalized for screen  $E$ .

## Question 9

[Solution n°25 p 49]

Carefully represent the previous function.

## Question 10

[Solution n°26 p 50]

What is the value of the interfringe?

## Question 11

[Solution n°27 p 50]

For which values of the optical path difference is there contrast cancellation?

## Question 12

[Solution n°28 p 50]

A source emits a spectral line with an energy spectral density function that can be described by a gaussian shape:

$$S_s(\sigma) = I_0 \exp\left(-\kappa \frac{(\sigma - \sigma_0)^2}{\Delta \sigma^2}\right)$$

where  $\kappa = -4 \ln 2$ ,  $\sigma_0$  is the average wave number. The line is centered on the wavelength where  $\Delta \sigma$  is the width at mid-height  $\lambda_0 = 546.1 \text{ nm}$ . The source has a spectral width of  $\Delta \lambda = 50 \text{ nm}$ .

What is the value of  $\Delta \sigma$  ?

## Question 13

[Solution n°29 p 50]

Find the normalized illumination for screen E.  
Calculate the coherence length of the source.

## Question 14

[Solution n°30 p 51]

Carefully graph the normalized illumination.

## Question 15

[Solution n°31 p 52]

What is the value of the interfringe distance?

## Question 16

[Solution n°32 p 52]

Re-do the previous questions for the case of a two-wavelength sodium source, taking into account the physical nature of each line: the fact that each line is described by a gaussian shape. Assume  $\Delta \lambda_0 \approx 0.002 \text{ nm}$  as the width of each line, which corresponds to 300K of inhomogeneous broadening due to the Doppler effect, and  $\Delta \lambda = 0.6 \text{ nm}$  for the distance between the two lines.

What are the values of  $\Delta \sigma_0$  and  $\Delta \sigma$  ?

## Question 17

[Solution n°33 p 52]

Recalculate the normalized illumination considering that each sodium doublet can be described by the Gaussian shape.

Graph the spectrum of the source and the obtained interferogram.

# Solution des exercices

## >Solution n°1 (exercice p. 39)

True.

## >Solution n°2 (exercice p. 39)

False, they are not related. However, the longer the lifespan of the wave train, the more the source is monochromatic and the narrower the spectrum.

## >Solution n°3 (exercice p. 39)

True,  $l_c = c\tau_c$

## >Solution n°4 (exercice p. 39)

In vacuum  $(\vec{E}, \vec{B}, \vec{K})$ , form an orthogonal triad; the statement, therefore, is true for  $(\vec{k}, \vec{E}, \vec{B})$  and  $(\vec{B}, \vec{k}, \vec{E})$ .

## >Solution n°5 (exercice p. 39)

True, according to Malus's theorem, the rays that form the energy trajectory are perpendicular to the wave surfaces.

## >Solution n°6 (exercice p. 39)

False, the waves are perpendicular to the wave surfaces.

## >Solution n°7 (exercice p. 39)

False, it is a narrow spectral line.

## >Solution n°8 (exercice p. 39)

True, in this case if we take  $\lambda_0 = 633 \text{ nm}$  (HeNe laser) and a lorentzian spectrum, the order of width is:  $l_c = \frac{0.318\lambda^2}{\Delta\lambda} = \frac{0.318 \times 633^2}{10^{-6}} \simeq 127 \text{ m}$

## >Solution n°9 (exercice p. 39)

False, a wave surface is the locus of points where the electrical field vectors oscillate strictly in phase. Vectors with a phase differing by  $2\pi$  belong to 2 different wave surfaces distant from  $\lambda$ .

## >Solution n°10 (exercice p. 39)

True for the same reason as in the previous question.

**>Solution n°11** (*exercice p. 39*)

No because two perpendicularly polarized waves do not interfere.

**>Solution n°12** (*exercice p. 40*)

Yes, but the average intensity of the interference figure is weaker than if the polarizations were parallel.

**>Solution n°13** (*exercice p. 40*)

True, in order for the contrast to be the greatest, the complex amplitudes associated to each source must be equal.

**>Solution n°14** (*exercice p. 40*)

The bright maxima corresponding to the orders of  $k = \{-4, -3, -2, -1, 0, +1, +2, +3, +4\}$  are located between the fifth minima situated on both sides of the fringe to the order of 0. We can count 9 fringes on the screen.

By assuming that  $\Delta x = 34.73 \text{ mm}$ , we can write the relation:

$$\Delta x = 9i = 9 \frac{\lambda d_0}{a}$$

from which:

$$\lambda = \frac{0.2 \times 34.73}{9 \times 1500} = 514 \times 10^{-4} \text{ mm} = 514 \text{ nm}$$

**>Solution n°15** (*exercice p. 40*)

At point **M** the optical path difference  $\delta$  is :

$$\delta = \frac{ax}{d_0} = \frac{1.25 \times 3}{1500} = 2.5 \mu\text{m}$$

The coherence length relative to the white source used (uniform source model) has a magnitude of ( $\lambda_0 = 550 \text{ nm}$  and  $\delta\lambda = 300 \text{ nm}$ ):

$$l_c = \frac{\lambda^2}{\Delta\lambda} = \frac{(550 \times 10^{-9})^2}{300 \times 10^{-9}} \approx 1 \mu\text{m}$$

The optical path difference is greater than the coherence length; consequently, the interference phenomenon can no longer be observed. Point **M** is located within the higher order white. By observing with a spectroscope, we can see that some wavelengths are missing.

**>Solution n°16** (exercice p. 40)

The dark lines in the spectrum correspond to destructive interferences. Therefore, the orders of interference are verifying:  $\delta = (2k + 1)\lambda/2$ .

They are included between the two extreme orders,  $k_B$  and  $k_R$  correspond to the two extremes of the spectrum. For the wavelength interval between  $\lambda_B = 400 \text{ nm}$  and  $\lambda_R = 700 \text{ nm}$ , the extreme values of the orders of interference are:

$$k_B = \frac{ax}{\lambda_B d_0} - \frac{1}{2} = \frac{1.25 \times 3}{0.4 \times 10^{-3} \times 1500} - \frac{1}{2} = \frac{2.5}{0.4} - \frac{1}{2} = 5.75$$

and

$$k_R = \frac{ax}{\lambda_R d_0} - \frac{1}{2} = \frac{1.25 \times 3}{0.7 \times 10^{-3} \times 1500} - \frac{1}{2} = \frac{2.5}{0.7} - \frac{1}{2} = 3.07$$

Since the orders of interference are not integers, there are only two dark lines in the spectrum that correspond to the orders  $k_1 = 4$  and  $k_2 = 5$ , and therefore to the wavelengths:

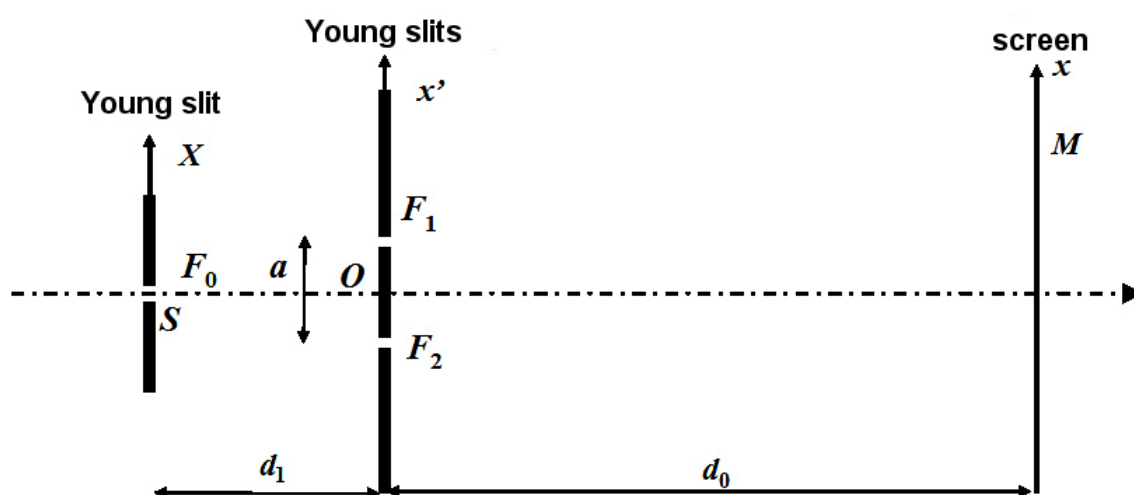
$$\lambda_1 = \frac{2\delta}{2k_1 + 1} = \frac{2 \times 2.5}{9} = 0.5556 \mu\text{m} = 555.6 \text{ nm}$$

and

$$\lambda_2 = \frac{2\delta}{2k_2 + 1} = \frac{2 \times 2.5}{11} = 0.4545 \mu\text{m} = 454.5 \text{ nm}$$

**>Solution n°17** (exercice p. 40)

The set-up of the model and the notations used are shown in figure 25.



Let us take the case of a two wave interferometer with perfectly coherent light, since the source is punctual and monochromatic, the distribution of illumination is:

$$I = \frac{I_0}{2} \left( 1 + \cos \left( \frac{2\pi\delta}{\lambda} \right) \right)$$

In the case of Young's slits, the interfringe distance is expressed as:

$$i = \frac{\lambda d_0}{a} = \frac{550 \times 10^{-9} \times 1}{2 \times 10^{-3}} = 0.275 \text{ mm}$$

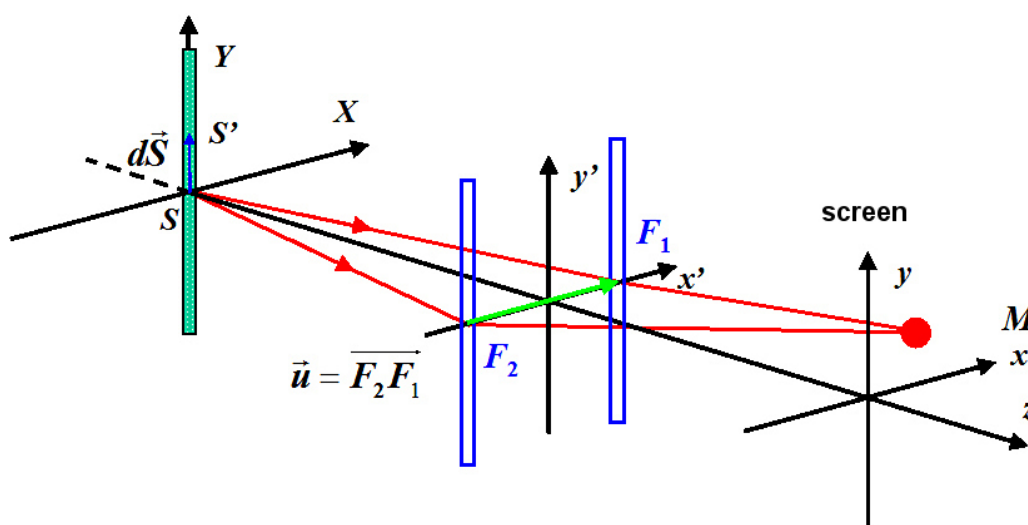
Since the source is perfectly coherent, the maximum contrast is equal to 1.

### >Solution n°18 (exercice p. 40)

If the source  $S$  is replaced by an infinitely narrow vertical slit, the different source points are incoherent amongst themselves and the light is partially spatially coherent. The optical path difference for a point  $M$  on the screen is given by the relation in paragraph D.3 of this course (**Evaluation of the Optical Path Difference**):

$$\delta(S') = \delta_0 + \frac{1}{d_1} (\vec{u}_1 - \vec{u}_2) \cdot d\vec{S}$$

Here the vectors  $\vec{u}_1 - \vec{u}_2 = \overrightarrow{F_2 F_1}$  and  $d\vec{S}$  are orthogonal (figure 26).



Consequently  $(\vec{u}_1 - \vec{u}_2) \times d\vec{S} = 0$  and the degree of coherence is equal to 1, the contrast does not diminish.

### >Solution n°19 (exercice p. 41)

A given fringe corresponds to a given optical path difference  $\delta$ . If we interpose a glass plate in the trajectory of  $F_1$  (1) the optical path increases because the index of the plate is superior to the index of the air. In order to compensate for this difference, the path length of the trajectory (2) must increase so that the value of the optical path difference does not change.

1. Therefore, the fringes pass through the side where the plate has been inserted.

2. The 2.9 mm shift is caused by the passing of the fringes:

$$k = \frac{t}{i} = \frac{2.9}{0.275} = 10.54 \quad \text{because a spatial period corresponds to an interfringe distance.}$$

3. When the glass plate is interposed, the optical path difference varies because on a thickness of  $e$  the air index 1 is replaced by the index  $n$  of the glass. The variation is thus:  $\Delta\delta = (n-1)e$

The passage from one fringe to the next of the same nature (dark or bright) corresponds to a variation in the optical path difference of  $\lambda$ , from which:

$$e = \frac{k\lambda}{n-1} = \frac{10.54 \times 0.550}{1.5-1} = 11.6 \mu\text{m}$$

### >Solution n°20 (exercice p. 41)

Both slits do not have the same intensity, so we must split the current relation:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\frac{2\pi\delta}{\lambda}\right)$$

The relation between both intensities verifies:

$$\text{DO} = -\log_{10}\left(\frac{I_1}{I_2}\right)$$

from which:

$$I_2 = 100 I_1$$

i.e.:

$$I = I_1 \left[ 101 + 20 \cos\left(\frac{2\pi\delta}{\lambda}\right) \right]$$

which produces a contrast of:

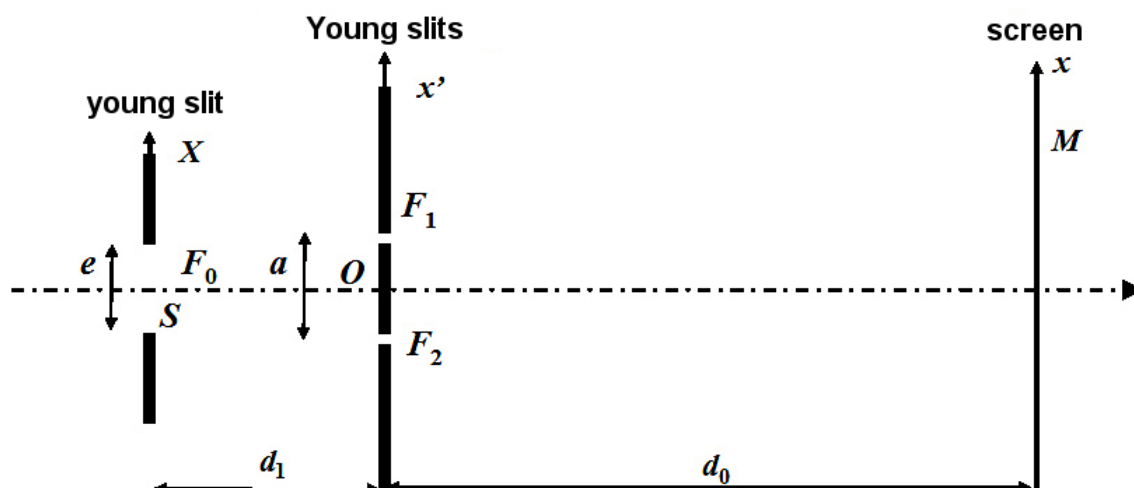
$$C = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{121 - 81}{121 + 81} = 0.2$$

### >Solution n°21 (exercice p. 41)

The case of a white light source is the same as the one described in the case study in paragraph II.B "**Case of white light**"

### >Solution n°22 (exercice p. 41)

The sources still have the same intensity, but the source is no longer spatially coherent because it has a width of  $e$ . The set-up and notations used are represented in 27.



The interference fringes have a weaker contrast. The intensity distribution on the screen is given by:

$$I = \frac{I_0}{2} \left( 1 + \Re \left[ \frac{\tilde{L}(\sigma_0 \alpha, \sigma_0 \beta)}{\tilde{L}(0,0)} \right] \cos(2\pi \sigma_0 \delta_0) \right)$$

$\tilde{L}$  represents the Fourier transform of the source's brightness function.

The source's spectral distribution of luminance can be described by a rectangle function with a width of  $e$  :

$$L(X, Y) = \Pi_e(X)$$

for which the Fourier transform is:

$$\tilde{L}(\sigma_0 \alpha, 0) = e \operatorname{sinc}(e \sigma_0 \alpha)$$

where  $\sigma_0$  is the wave number of the source and  $\alpha = a/d_1$  is the angle from which the slits are seen from the source center.

Consequently, the degree of spatial coherence is equal to:

$$\gamma_{c,s} = \Re \left[ \frac{\tilde{L}(\sigma_0 \alpha, 0)}{\tilde{L}(0,0)} \right] = \operatorname{sinc} \left( \frac{ea}{\lambda_0 d_1} \right) = \operatorname{sinc} \left( \frac{2 \times 6.8 \times 10^{-3}}{0.555 \times 10^{-3} \times 100} \right) = 0.9$$

### >Solution n°23 (exercice p. 41)

First, let's consider a single source slit that has been shifted on the plane  $(X, Y)$  by a quantity  $p$  following the  $X$  axis. For a point  $M$  on the screen the optical path difference varies by  $\Delta\delta = ap/d_1$ . The calculation is analogous to the one in the Case Study. If this optical path difference is equal to  $k\lambda$  then the figure of interference is identical to the preceding one. The figure is translated downwards by the equivalent of one interfringe distance. A bright fringe (respectively dark) has replaced the following bright fringe (respectively dark).

Consequently, the single source slit can be replaced by a lattice:

$$p = \frac{\lambda d_1}{a} = \frac{0.550 \times 100}{2} = 27.5 \mu\text{m}$$

### >Solution n°24 (exercice p. 41)

The source is now spatially coherent, but partially temporally incoherent.

As seen in paragraph C.3.b ("Examples" in **Temporal Coherence**), the energy spectral density is expressed as:

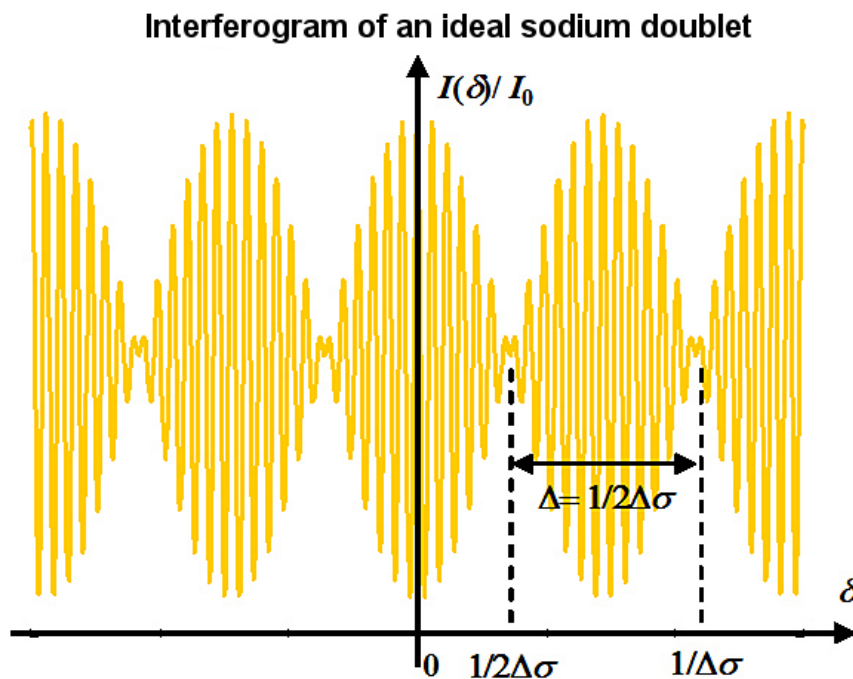
$$S_s(\sigma) = I_0 \delta\left(\sigma - \sigma_0 - \frac{\Delta\sigma}{2}\right) + I_0 \delta\left(\sigma - \sigma_0 + \frac{\Delta\sigma}{2}\right)$$

and we have (see paragraph C.3.b "**Examples**"):

$$I(\delta) = \frac{I_0}{2} \left[ 1 + \cos(\pi \Delta\sigma \delta) \cos(2\pi \sigma_0 \delta) \right]$$

### >Solution n°25 (exercice p. 41)

Graph of the intensity distribution on the screen: the interferogram has beats and a similar structure to the one shown on figure 28.



**>Solution n°26** (exercice p. 41)

The interferogram of the doublet is also written as:

$$I(\delta) = \frac{I_0}{2} \left[ 1 + \cos(\pi \Delta \sigma \delta) \cos\left(2\pi \frac{\delta}{\lambda_0}\right) \right]$$

The inferfringe distance is thus given by:

$$i = \frac{\lambda_0 d_0}{a}$$

with  $\lambda_0 = 1/\sigma_0 = 2\lambda a_1 \lambda_2 / (\lambda_1 + \lambda_2) = 589.3 \text{ nm}$ ,

Therefore, we have:

$$i = \frac{0.5893 \times 10^{-3} \times 1000}{2} = 0.294 \text{ mm}$$

**>Solution n°27** (exercice p. 41)

The contrast function  $\gamma_{c, i} = \cos(\pi \Delta \sigma \delta)$  cancels out when the cosine function is equal to  $\pi/2 + k\pi$ , i.e.:

$$\delta = \frac{1}{2\Delta\sigma} + \frac{k}{\Delta\sigma}$$

**>Solution n°28** (exercice p. 42)

Same procedure as the previous exercise.

The value of  $\Delta\sigma$  is given by:

$$\Delta\sigma = \frac{\Delta\lambda}{\lambda^2} = \frac{50 \times 10^{-9}}{(546.1 \times 10^{-9})^2} = 1.676 \times 10^3 \text{ cm}^{-1}$$

**>Solution n°29** (exercice p. 42)

The energy spectral density is:

$$S_s(\sigma) = I_0 \exp\left(-\kappa \frac{(\sigma - \sigma_0)^2}{\Delta\sigma^2}\right)$$

The spectral shape is thus:

$$S_0(\sigma) = I_0 \exp\left(-\kappa \frac{\sigma^2}{\Delta \sigma^2}\right)$$

The Fourier transform of  $S_0(\sigma)$  is:

$$\tilde{S}_0(\delta) = \frac{\sqrt{\pi} \Delta \sigma^2}{\sqrt{\kappa}} I_0 \exp\left(-\left(\frac{\pi \Delta \sigma \delta}{\sqrt{\kappa}}\right)^2\right)$$

The degree of coherence is:

$$\gamma_{c,t}(\delta) = \Re \left[ \frac{\tilde{S}_0(\delta)}{\tilde{S}_0(0)} \right] = \exp\left(-\frac{1}{\kappa} \pi^2 \Delta \sigma^2 \delta^2\right)$$

The interference signal is expressed as:

$$\frac{I(\delta)}{I_0} = \frac{1}{2} \left[ 1 + \exp\left(-\frac{1}{\kappa} \pi^2 \Delta \sigma^2 \delta^2\right) \cos(2\pi \sigma_0 \delta) \right]$$

The coherence length of the source can be calculated by using the expression:

$$l_c = \int_{-\infty}^{+\infty} |\gamma_{c,t}(\delta)|^2 d\delta$$

i.e.:

$$l_c = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{\kappa} \pi^2 \Delta \sigma^2 \delta^2\right) d\delta$$

Taking into account that:

$$\int_{-\infty}^{+\infty} \exp(-x^2) dx = \sqrt{\pi}$$

the result is:

$$l_c = \frac{0.664}{\Delta \sigma} = 0.664 \frac{\lambda^2}{\Delta \lambda}$$

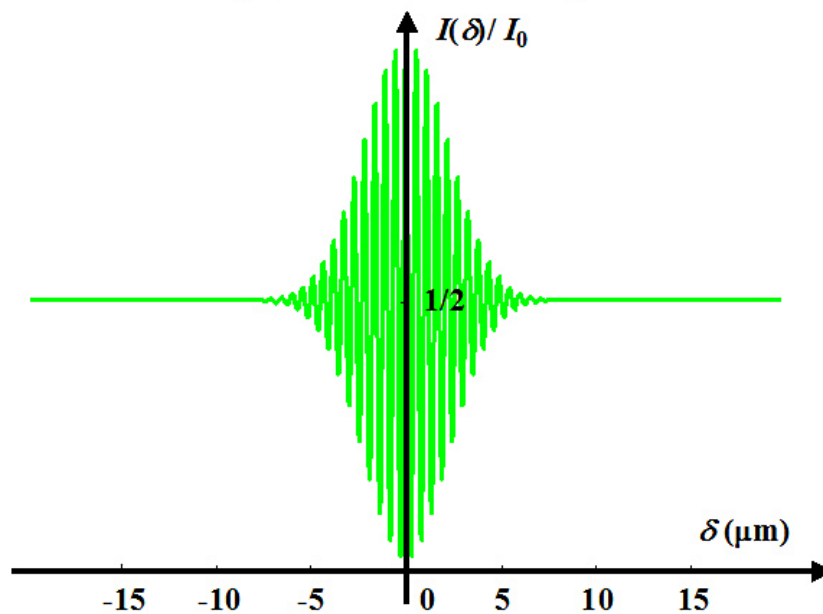
When this value is applied it produces:

$$l_c = \frac{0.664}{1.6766 \times 10^3} = 3.96 \times 10^{-4} \text{ cm} = 3.96 \mu\text{m}$$

### >Solution n°30 (exercice p. 42)

The standard illumination has an average value of  $1/2$ . It is shown on figure 29.

## Interferogram of a Gaussian source



## &gt;Solution n°31 (exercice p. 42)

The cosine function always has the same period  $1/\sigma_0 = \lambda_0$ , the geometry does not change so the period on the screen and the interfringe remain the same.

## &gt;Solution n°32 (exercice p. 42)

The value of  $\Delta\sigma_0$  is given by:

$$\Delta\sigma_0 = \frac{\Delta\lambda_0}{\lambda^2} = \frac{0.005 \times 10^{-7}}{(589.3 \times 10^{-7})^2} = 0.057 \text{ cm}^{-1}$$

and the value of  $\Delta\sigma$  is given by:

$$\Delta\sigma = \frac{\Delta\lambda}{\lambda^2} = \frac{0.6 \times 10^{-7}}{(589.3 \times 10^{-7})^2} = 17.277 \text{ cm}^{-1}$$

## &gt;Solution n°33 (exercice p. 42)

When each one of the sodium doublets can be described by a Gaussian function, the case is similar to the real situation. The energy spectral density function is written as:

$$S_S(\sigma) = I_0 \exp\left(-\kappa \frac{(\sigma - \sigma_0)^2}{\Delta\sigma_0^2}\right) * \left[ \delta\left(\sigma - \left(\sigma_0 + \frac{\Delta\sigma}{2}\right)\right) + \delta\left(\sigma - \left(\sigma_0 - \frac{\Delta\sigma}{2}\right)\right) \right]$$

The procedure is the same as in the previous exercise. We calculate the Fourier transform of the spectral energy density, which is equal to the product of the Fourier transforms of the two convoluted functions.

These Fourier transforms have been previously calculated.

The two Dirac distributions convoluted to the gaussian function are written as:

$$f(\sigma) = \delta\left(\sigma - \sigma_0 - \frac{\Delta\sigma}{2}\right) + \delta\left(\sigma - \sigma_0 + \frac{\Delta\sigma}{2}\right)$$

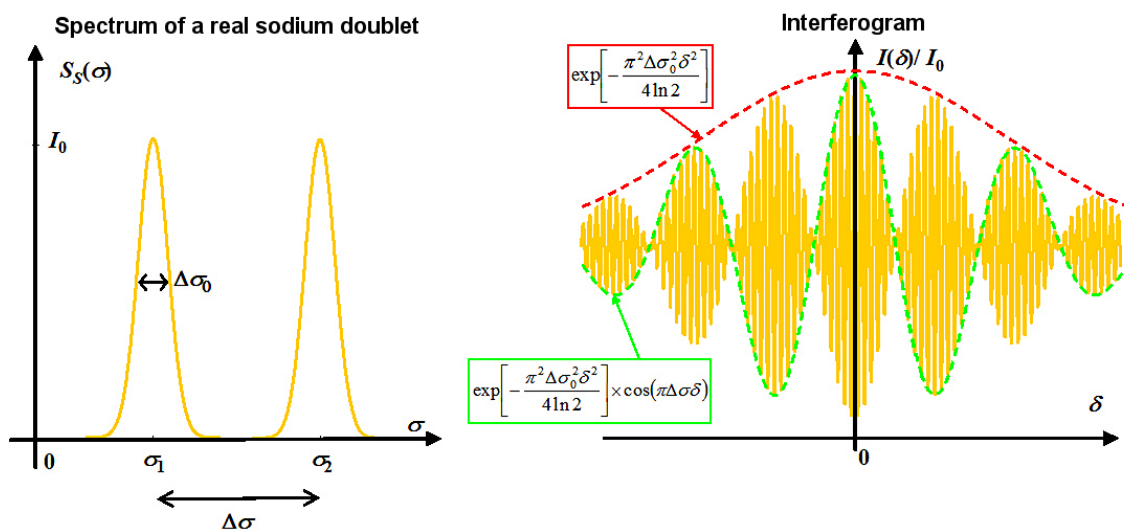
and their Fourier transform is:

$$\tilde{f}(\delta) = \cos(\pi \Delta \sigma \delta)$$

From this, we can deduce the expression of the normalized interferogram:

$$\frac{I(\delta)}{I_0} = \frac{1}{2} \left[ 1 + \exp\left(-\frac{1}{K} \pi^2 \Delta \sigma_0^2 \delta^2\right) \cos(\pi \Delta \sigma \delta) \cos(2 \pi \sigma_0 \delta) \right]$$

The intensity as a function of the optical path difference is represented in figure 30.



# Bibliographie

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