Fractional Fourier Transform

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I.Présentation

Module :

Interference and Diffraction

Auteur(s) :

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Résumé :

The main objective of the chapter is to define Fractional Fourier Transforms so as to acess to a so-called fractional domain between the spatial and the spatial-frequency domains through mere operators whih are closely linked to time-frequency distributions. Finally, a digital way is offered to calculate the result of this operation.

Mots-clés :

Fourier transform, Decomposition, Time-frequency, Fractional

Pré-requis :

Fourier transforms, decomposition of a funtction

Objectif(s) pédagogique(s) :

Prove that we can choose a description domain different from those known to use (time, space or frequency)

Plan du cours :

- Introduction
- Fractional Fourier Transformation
- Fractional Fourier transformation and Wigner and Radon distributions
- Differential equation solving
- Digital fractional Fourier Transforms

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II.Lesson

The main objective of this lesson is to show the elements necessary to understand Fractional Fourier Transforms concisely and precisely. Two notions appear in our title: 1) Fourier transforms and 2) Fractional order. We do not need to indicate the major implications of the first subject in physics, more specifically its implication on Fourier physical optics and signal processing. The fractional order has nothing to do with fractional calculations. But, for the time being, this underlines a fundamental question, that is if the description domain of a signal exists between the spatial and spectral domains. Fractional Fourier Transforms are a generalization of Fourier Transforms. When the conventional Fourier Transform of a function f can be written $\mathsf{F}_{\pi/2}f$, this writing can be generalized for any fractional order α . The, α-order Fractional Fourier Transform ($FRFT$) of f will be:

$$\mathsf{F}_{\alpha} f$$
$$(1.1)$$

With $\alpha = 0$, the identity operator can be found and the result will be the function f. With $\alpha = \pi/2$, the conventional Fourier Transformation is found. We must keep in mind that the main objective is to define Fractional Fourier Transformation so as to access to a so-called fractional domain between the spatial and the spectral domains. We must remember that time-frequency bilinear operators also exist. These operators are numerous and their main objective is to give a signal representation on a map of spatial-frequency or time-frequency coordinates. We will look into Wigner and Radon-Wigner representations.

We will examine the following elements :

1. The mathematical definition of fractional Fourier Transformation ,
2. The properties linked to this operator ,
3. Bilinear representations and fractional Fourier Transformation ,
4. Digital Fractional Fourier Transformation

1. Fractional Fourier Transformation

1.1. Definition

Within the framework of the solving of differential equations in quantum mechanics, Namias proposed a new transformation, the FRactional Fourier Transform ($FRFT$) [[The Fractional order Fourier Transform and its application to Quantum]]. This transformation is based on the following property: Normalized Hermite-Gauss functions, written $\phi_n(x)$ where $n \in \mathbb{N}$ is the function order, are specific FT functions linked to the specific values $\mathrm{Exp}\,(-in\frac{\pi}{2})$ such that:

$$\mathsf{F}[\Phi_n(x)](u) = \exp(-i\,n\frac{\pi}{2})\Phi_n(u)$$
$$(2.1)$$

with $i^2 = -1$.F is the conventional Fourier transformation operator defined by

$$\mathsf{F}[f(x)](u) = \int_{-\infty}^{+\infty} f(x)\exp(-i\,2\,\pi\,ux)\,dx$$
$$(2.2)$$

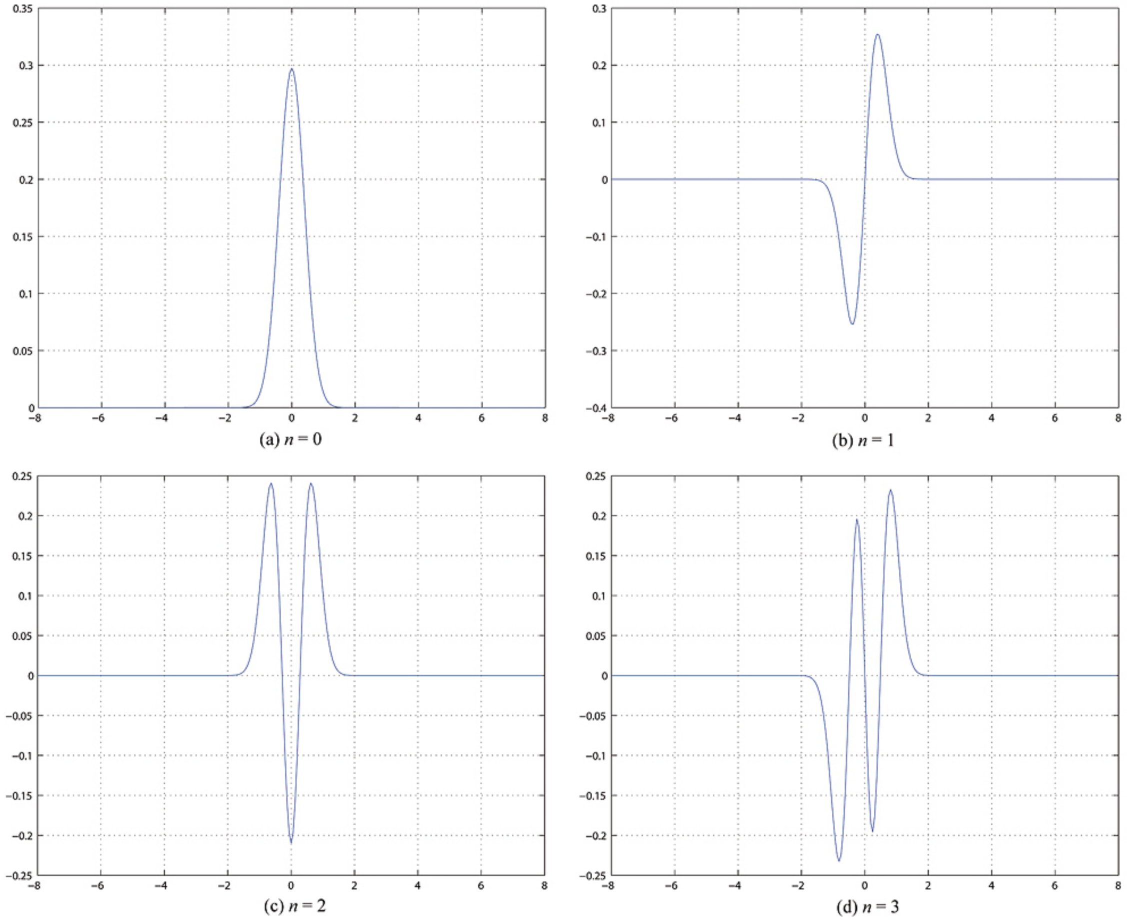
and

$$\Phi_n(x) = \frac{2^1/4}{\sqrt{2^n n!}} . h_n(x\sqrt{2\pi}) \exp(-\pi x^2)$$
$$(2.3)$$

The function $h_n(x)$ represents the Hermite polynomial of order n. Its value is given by:

$$h_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2)$$
$$(2.4)$$

Hermite-Gauss functions are real and oscillating functions. The order n indicates the number of roots of the function $\phi_n(x)$ as shown in figure 1.



(a) $n = 0$ (b) $n = 1$

(c) $n = 2$ (d) $n = 3$

Figure 1: Hermite-Gauss functions of orders $0, 1, 2$ *et* 3

As the phase of proper values in (2.1) is proportional to $\frac{\pi}{2}$, Namias called Fourier Operator $F_{\frac{\pi}{2}}$. Then, he generalized this operator to any fractional order $\alpha \in \mathbb{R}$ such that:

$$F_\alpha[\Phi_n(x)](u) = \mathrm{Exp}[-\mathrm{i}n\alpha]\Phi_n(u)$$
$$(2.5)$$

The operator F_α is thus called the fractional Fourier Transformation of order α where α is the fractional order. It should be noted that $\mathsf{F}_{\pi/2}$and F_α are not the same operators. We must keep in mind that Hermite-Gauss functions form an orthonormal basis [[Elements de mathématiques du signal]]. Accordingly, any square-integrable function f can be factorized on this basis as follows:

$$f(x)=\sum_{n=0}^{\infty} a_n \Phi n(x) \tag{2.6}$$

where a_n are the decomposition coefficients such that:

$$a_n=\frac{1}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \Phi_n(x) f(x) dx \tag{2.7}$$

By taking into account (2.5), a first definition of the $FRFT$ of $f(x)$ can be given by:

$$\mathsf{F}_\alpha[f(x)]=\sum_{n=0}^{\infty} a_n \mathrm{Exp}(-\mathrm{in}\,\alpha) \mathrm{Exp}(-\frac{x^2}{2}) h_n(x) \tag{2.8}$$

Still, the theoretical calculation of the $FRFT$ of a function through a series is not easy. An integral writing was given by using the Mehler formula [[The Fractional order Fourier Transform and its application to Quantum]] The Fractional order Fourier Transform and its application to Quantum. (2.8) then becomes:

$$\mathsf{F}_\alpha[f(x)](x_a)=C(\alpha) \exp(i \pi \cot \alpha\, x_a^2) \int_{-\infty}^{+\infty} f(x) \exp(i \pi \cot \alpha\, x^2) \exp(-i 2 \pi \frac{x_a x}{\sin \alpha}) dx \tag{2.9}$$

with

$$C(\alpha)=\frac{\exp(-i(\frac{1}{4} \pi\, sign(\sin \alpha)-\alpha/2))}{\sqrt{|\sin \alpha|}} \tag{2.10}$$

$\forall \alpha \in \mathbb{R}$. x_a is the variable in the fractional domain. The term $C(\alpha)$ ensures the energy conservation between the spatial and the fractional domains. Eq. (2.9) can be written in a simpler way:

$$\mathsf{F}_\alpha[f(x)](x_a)=\int_{-\infty}^{+\infty} \mathsf{K}_\alpha(x, x_a) f(x) dx \tag{2.11}$$

Where $\mathsf{K}_\alpha(x, x_a)$ is the kernel of the $FRFT$. It equals to:

$$\mathsf{K}_\alpha(x, x_a) = \sum_{n=0}^{\infty} \Phi_n(n) \exp(-i n \alpha) \Phi_n(x_a) = C(\alpha) \exp(i \pi \cot \alpha (x^2 + x_a^2)) \exp(-i 2 \pi \frac{x_a x}{\sin \alpha})$$

$$\text{if } \alpha \text{ is not a multiple of } \pi.$$

$$= \delta(x - x_a)$$

$$\text{if } \alpha \text{ is not a multiple of } 2\pi.$$

$$= \delta(x + x_a)$$

$$\text{if } \alpha + \pi \text{ is a multiple of } 2\pi.$$

$$(2.12)$$

The symbol $\delta(x)$ represents the Dirac distribution. The $FRFT$ definition is valid for any function $f(x)$, whether square integrable or not ($1, x, x^2, \ldots$), provided that the integral is convergent. Finally, it should be noted that the fractional order α can have a complex value.

1.2. The intrinsic properties of fractional operators

Among the properties of $FRFTs$ [[The Fractional Fourier Transform]], the identity operator corresponds to a fractional order equal to 0: $\mathsf{F}_0[f(x)] = f(x)$. The parity of a signal is obtained when $\alpha = \pi$. Indeed $\mathsf{F}_\pi[f(x)] = f(-x)$. The particular case of conventional Fourier transformation is obtained when the fractional order equals to $\frac{\pi}{2}$:

$$\mathsf{F}_{\frac{\pi}{2}}[f(x)] = \mathsf{F}[f(x)]$$

$$(2.13)$$

The inverse fractional Fourier transform of a signal is obtained by its direct transformation by changing the sign of α:

$$f_a(x_a) = \mathsf{F}_\alpha[f(x)](x_a) \Leftrightarrow f(x) = \mathsf{F}_{-\alpha}[f_a(x_a)](x)$$

$$(2.14)$$

Parseval's relation is preserved between the time or spatial domain and the fractional domain:

$$\int_{-\infty}^{+\infty} [f(x)]^2 dx = \int_{-\infty}^{+\infty} [\mathsf{F}_\alpha(f(x))]^2 (x_a) dx_a$$

$$(2.15)$$

The $FRFT$ is a linear operator, i.e.:

$$\mathsf{F}_\alpha[\sum_n c_n f_n(x)](x_a) = \sum_n c_n \mathsf{F}_\alpha[f_n(x)](x_a)$$

$$(2.16)$$

We must keep in mind that the translation of the operator is non-linear. If $f(x-\zeta)$is the translated version of $f(x)$ of the quantity ζ, its transformed version will be:

$$\mathsf{F}_{\alpha}[f(x-\zeta)](x_a)=\exp(i\pi\zeta^2\sin\alpha\cos\alpha)\exp(-i2\pi\zeta\sin\alpha)F_{\alpha}[f(x)](x_a-\zeta\cos\alpha)$$
$$(2.17)$$

Non-linearity is translated by the coefficient $cos\alpha$ on ζ. This non-linearity disappears as soon as we are in Fourier's domain, i.e. when $\alpha=\pi/2$. This remark is in accordance with what we know about Fourier transformation: this operator is invariant when translated. The composition of two operators of fractional orders α and β is equal to the sum of the two fractional orders such that:

$$\mathsf{F}_{\alpha}\circ\mathsf{F}_{\beta}=\mathsf{F}_{\beta}\circ\mathsf{F}_{\alpha}=\mathsf{F}_{\alpha+\beta}.$$
$$(2.18)$$

Finally the $FRFT$ of a function multiplied by a complex exponential is:

$$\mathsf{F}_{\alpha}[\exp(i2\pi\xi x)f(x)](x_a)=\exp(-i\pi\xi^2\sin\alpha\cos\alpha)\exp(i2\pi x\xi\cos\alpha)\mathsf{F}_{\alpha}[f(x)](x_a-\xi\sin\alpha)\cdot$$
$$(2.19)$$

The associative and commutative properties of the operator regarding the fractional order α can be added to these properties. The fractional Fourier transform of the derivative of order n of a function $f(x)$ is defined by:

$$\mathsf{F}_{\alpha}[\frac{d^n f}{dx^n}(x)](x_a)=(i2\pi x_a\sin\alpha+\cos\alpha\frac{d}{dx_a})^n f_a$$
$$(2.20)$$

with $f_a=\mathsf{F}_{\alpha}[f(x)](x_a)$. Two examples can be given. The first one when $n=1$:

$$\mathsf{F}_{\alpha}[\frac{df}{dx}(x)](x_a)=(i2\pi x_a\sin(\alpha)f_a+\cos(\alpha)\frac{df_a}{dx_a})$$
$$(2.21)$$

The second one when $n=2$:

$$\mathsf{F}_{\alpha}[\frac{d^2}{dx^2}f(x)](x_a)=[-4\pi x_a^2\sin^2\alpha+i\pi\sin(2\alpha)]f_a+i2\pi\sin(2\alpha)x_a\frac{df_a}{dx_a}+\cos^2\alpha\frac{d^2f_a}{dx_a^2}$$
$$(2.22)$$

The last interesting property when solving a differential equation is the transform of the function $f(x)$ multiplied by x^n:

$$\mathsf{F}_{\alpha}[x^n f(x)](x_a)=(x_a\cos\alpha-\frac{\sin\alpha}{i2\pi}\frac{d}{dx_a})^n f_a$$
$$(2.23)$$

When $n = 1$, we obtain:

$$\mathsf{F}_\alpha[xf(x)](x_a) = x_a f_a \cos\alpha - \frac{\sin\alpha}{i2\pi}\frac{df_a}{dx_a}$$
$$(2.24)$$

and $n = 2$, we obtain:

$$\mathsf{F}_\alpha[x^2 f(x)](x_a) = \frac{\sin 2\alpha}{4\pi}[i + 2\pi x_a^2 \cot\alpha] f_a + \frac{i}{2\pi} x_a \sin(2\alpha)\frac{df_a}{dx_a} - \frac{\sin^2\alpha}{4\pi^2}\frac{d^2 f_a}{dx_a^2}$$
$$(2.25)$$

2. Fractional Fourier transformation and Wigner and Radon distributions

Usual tools for signal analysis, such as conventional Fourier transformation, cannot give a representation of the spectral evolution of a signal. That is why time-frequency representations are more and more used. These representations have the great advantage to jointly describe a signal in time and frequency. They give a natural description of non-stationary signals whose frequency change over time. These signals are often called “chirped signals” where the term “chirp” [[The Fractional order Fourier Transform and its application to Quantum]] refers to frequency derivatives. This type of description requires the use of a bilinear decomposition, i.e. decomposition in time and frequency [[Temps-Fréquence]]. A first class of solutions for such representations uses atomic decompositions (better known as linear time-frequency representations). Intuitively, we can analyze a signal with a short-time Fourier Transformation (or window Fourier Transformation). Wavelet transformation and Gabor's transformation are solutions too. Yet, they do not give a representation of energy distribution in time and frequency [[Temps-Fréquence]]. To obtain the energetic representation of a signal with the first class of solutions, the square module of short-time Fourier transformation or wavelet transformation are calculated to give the spectrogram or scalogram respectively. There is a second class of solutions defined by energy distribution following *the fundamental principle of covariance by time and frequency translation* such as Wigner or pseudo-Wigner-Ville distributions. This distribution is a particular case of a more general class called Cohen's Class [[Temps-Fréquence]]. Unlike linear representations which split a signal in elementary components, the goal of associated energy distribution is to allocate it among the two description variables. In other words, in Parseval's relation:

$$E_f = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(\nu)|^2 d\nu$$
$$(2.26)$$

$|f(x)|^2$ and $|F(\nu)|^2$ can be seen as energy density respectively in time and frequency. Therefore, it is normal to look for an energy density depending both on time and frequency to get the energy of the signal:

$$E_f = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho_f(x,\nu)\, dx\, d\nu$$
$$(2.27)$$

The associated energy density must meet two properties, namely:

$$|f(x)|^2=\int_{-\infty}^{+\infty}\rho_f(x,v)\,dv$$

$$|F(v)|^2=\int_{-\infty}^{+\infty}\rho_f(x,v)\,dx$$
$$(2.28)$$

Cohen's class was created under these conditions and following the fundamental principle of covariance by translation in time and frequency.

2.1. The Wigner Distribution Function

We must keep in mind that the variables are either time or space. We will write them t or x. The most general form of time-frequency energy distributions belonging to Cohen's Class is defined by the overall mathematical equation:

$$C_{f,g}(x,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Pi(x-s,v-\xi)W_{f,g}(s,\xi)\,dsd\,\xi$$
$$(2.29)$$

The function $\Pi(x,\nu)$ is a function called kernel that can be adapted to the chosen signal to lower the interference terms (as we will see it later) in a controlled way. This kernel can be seen as a smoothing window influencing time-frequency representations and $W_{f,g}$ is the cross-Wigner distribution of f and g :

$$W_{f,g}(x,v)=\int_{-\infty}^{+\infty}f\left(x+\frac{x'}{2}\right)g*\left(x-\frac{x'}{2}\right)\exp(-i2\pi vx')\,dx'$$
$$(2.30)$$

The equation. (2.29) is nothing but the convolution product of $\Pi(x,\nu)$ by $W_f(x,\nu)$. When $\Pi(x,\nu)=\delta(x,\nu)$, the distribution obtained is the same as the Wigner distribution function. The symbol $*$ refers to the conjugate expression. x and ν axes define an orthogonal basis. All the proprieties will not be explained here [[Theory and applications in signal processing]], but just those important for this lesson. The first one is the Wigner distribution of a conventional Fourier transformation: $F(\nu)=\mathcal{F}_{\pi/2}\left[f(x)\right](\nu)$. The Wigner distribution of $F(\nu)$ can be written following the Wigner distribution of $f(x)$ as follows:

$$W_F(x,v)=W_f(-v,x)$$
$$(2.31)$$

As we will see it later, this corresponds to a rotation of 90 in time-frequency space. If we change the frequency of the Wigner distribution with $g(x)=f(x)exp(i2\pi\chi x^2)$, we obtain a translation in time-frequency space:

$$W_g(x,v)=W_f(x,v-\beta)$$
$$(2.32)$$

The most important property in our case is the multiplication by a linear chirp function. If $g(x)=f(x)exp(i2\pi\beta x)$ so

$$W_g(x,\nu)=W_f(x,\nu+\beta x)$$
$$(2.33)$$

The last property is linked to the Wigner distribution of a multicomponent signal:

$$W_{af+bg}(x,\nu)=|a|^2 W_f(x,\nu)+|b|^2 W_g(x,\nu)+2\Re(ab^* W_{f,g}(x,\nu))$$
$$(2.34)$$

The first two terms well define the Wigner distribution functions f and g. The third term shows the non-linear property of the Wigner distribution. It is also called external interferences [[Theory and applications in signal processing]].

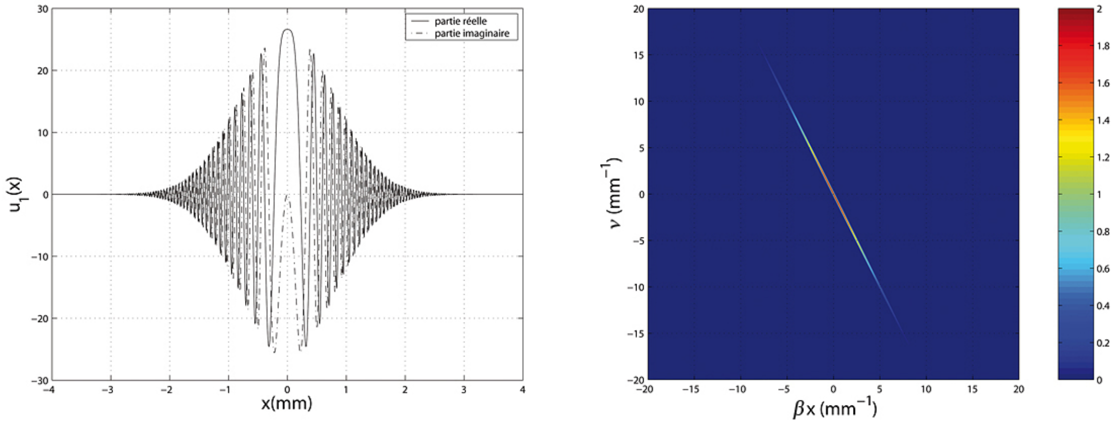


Figure 2 : Representation of $u_1(x)$ *and its associate Wigner distribution function.* $\sigma = 2\ mm$, $\beta = 5\ mm^{-2}$

2.2. Example of a Wigner distribution function

We will study the example of a linear chirp function written $u_1(x)$ such as:

$$u_1(x)=\left(\frac{2}{\sigma^2}\right)^{1/4}\exp\left(-\pi\frac{x^2}{\sigma^2}\right)\exp\left(-i2\pi\beta x^2\right)$$
$$(2.35)$$

Figure 2 illustrates the function $u_1(x)$ and its Wigner distribution function. We can see that the energy follows the frequency variation law ν according to the space variable x. The instantaneous frequency, written $\nu(x)$, is given by the equation [[Theory and applications in signal processing]]:

$$\nu(x)=\frac{1}{2\pi}\frac{\partial\phi(x)}{\partial x}$$
$$(2.36)$$

where $\phi(x)$ is the argument of the function $u_1(x)$. Eqs. (2.35) and (2.36) result in:

$$\nu(x)=-2\beta x$$
$$(2.37)$$

We can also define an angle ζ linked to the variation law of the instantaneous space frequency according to the variable βx, namely:

$$\tan \zeta = \frac{\nu(x)}{\beta x} = -2$$
$$(2.38)$$

So, $\xi = -63.43$. The slope of the Wigner distribution has an angle of -63.43 with the x axis. In this first example, where only one function is used, there are no interferences (third term of Eq. (2.34)). To illustrate the non-linearity of the function, we will add another component, written $u_2(x)$, to have a multicomponent signal:

$$u_2(x) = \left(\frac{2}{\sigma^2}\right)^{1/4} \exp\left(-\pi \frac{x^2}{\sigma^2}\right) \exp\left(-i \pi \left(2 \beta x^2 + \gamma x\right)\right)$$
$$(2.39)$$

with Eq. (2.34), we can write :

$$W_{u_1+u_2}(x,\nu) = W_{u_1}(x,\nu+2\beta x) + W_{u_2}\left(x,\nu+\frac{\gamma}{2}+2\beta x\right) + + \sum_{k=1\, k<l}^{N} \sum_{l=1}^{N} 2\Re\left[W_{u_k}, u_l\right]$$
$$(2.40)$$

In figure 3, two distinctive distributions related to $u_1(x)$ and $u_2(x)$ split in ν by $5\, mm^{-1}$ can be observed. Between these two distributions there are external interferences linked to the non-linearity of Wigner's operator. These interferences are characterized by positive and negative oscillations. In other words, if a signal consists of N elementary signals, then its Wigner distribution will consist of N Wigner distributions and ($\overset{N}{2}$) external interference terms. Other interferences can occur with unicomponent signals. For example, the function:

$$u_3(x) = \left(\frac{2}{\sigma^2}\right)^{1/4} \exp\left(-\pi \frac{x^2}{\sigma^2}\right) \exp\left(i 2 \pi \varepsilon x^3\right)$$
$$(2.41)$$

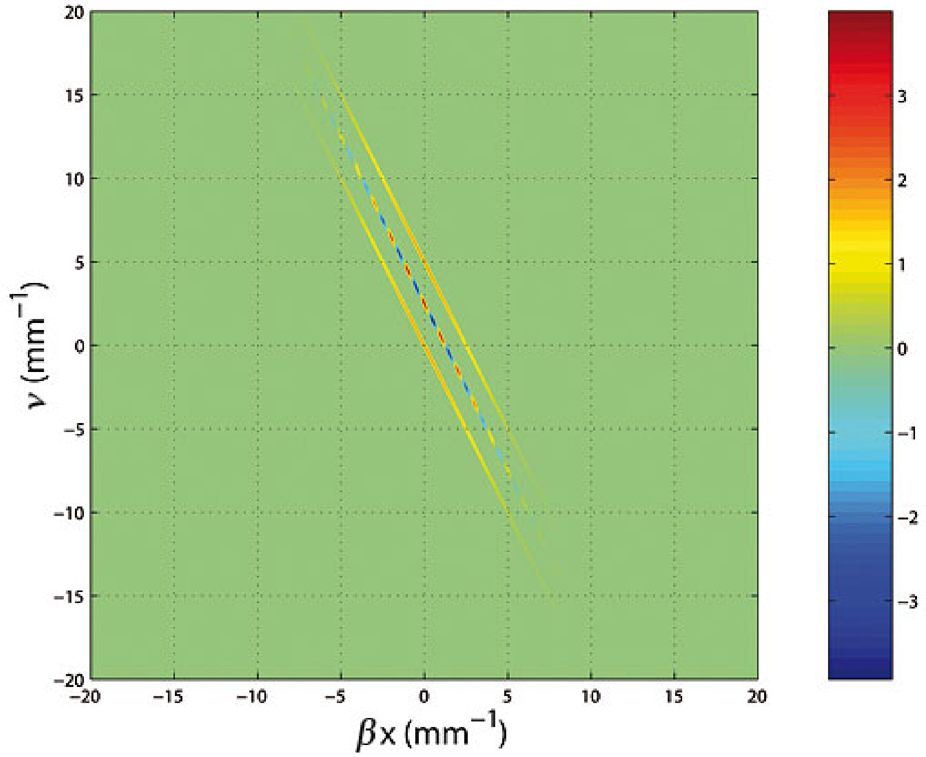


Figure 3: Representation of the Wigner distribution function of the multicomponent signal $u_1(x) + u_2(x)$ *with* $\sigma = 2\ mm$*,* $\beta = 5\ mm^{-2}$ *and* $\gamma = -10\ mm^{-1}$

whose Wigner distribution is given in figure 4. This distribution shows oscillations that are characteristic of the phenomenon of internal interferences. These interferences occur whenever the frequency variation law according to the space variable introduces a curvature radius (here $\nu(x) \simeq \epsilon x^2$). The equation (2.28) can be used to obtain the energy profile according to one of the two variables x or ν.

However, it would be interesting to calculate the energetic profile of the signal in a fractional domain between the spatial domain and the spatial frequency domain. With this in mind, the Radon transformation is the best suitable tool.

2.3. The Radon-Wigner transformation

This transformation was originally designed for medical diagnosis by X-ray digital tomography. With it, we could obtain data about the slices of the body [[Principles of Optics]]. The Radon transformation is obtained by the projection of the Wigner distribution function $W_f(x, \nu)$. Its projection along the x_μ axis makes an angle μ with the x axis (see figure 2.5). It is defined by:

$$\mathscr{R}_\mu[f](x_\mu)=\int_{-\infty}^{+\infty} W_f(x_\mu\cos\mu - v_\mu\sin\mu\,,\, x_\mu\sin\mu+v_\mu\cos\mu)\,dv_\mu \quad (2.42)$$

We will use the examples given in paragraph (B.2) ($u_1(x)$ and $u_2(x)$) and apply the Radon transform on the Wigner distribution by narrowing the angle range from 10 to 45. As shown in figure 6,

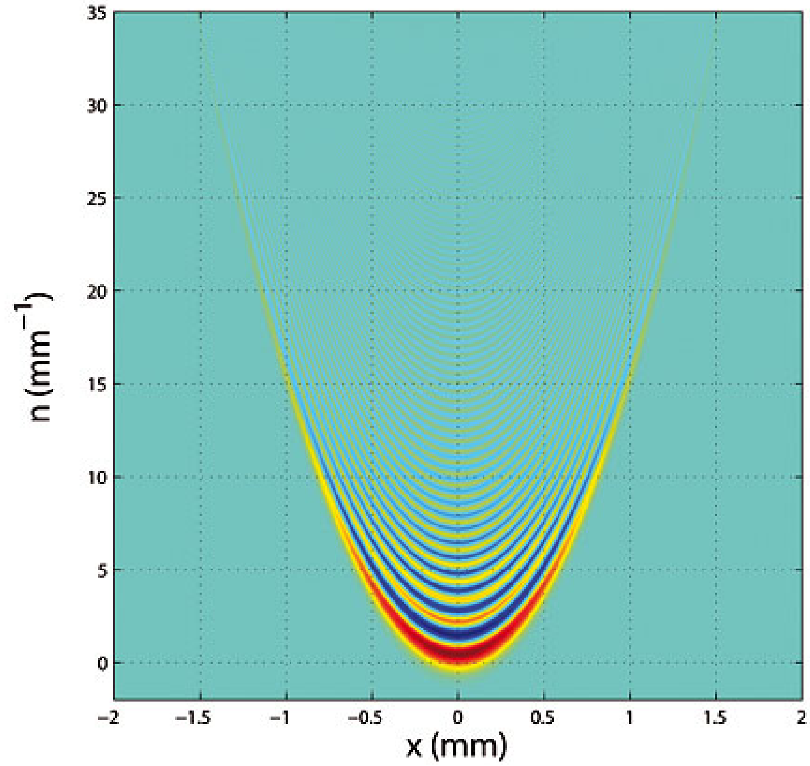


Figure 4 : Representation of the Wigner distribution function of the unicomponent signal $u_3(x)$ *with* $\sigma = 2\ mm$ *and* $\epsilon = 5\ mm^{-3}$

the two energy maxima associated with the two linear chirp functions $u_1(x)$ and $u_2(x)$ are observed at an angle that fits the complementary angle of ξ defined by Eq. (2.36):

$$\mu = \arctan\left(-\frac{1}{2}\right) \tag{2.43}$$

The value of μ obtained corresponds to the optimal angle of the Radon-Wigner transform to obtain a maximum of energy by projection: $|\mu| = 26.56$. However, it should be noted that this is not very efficient when we want to save computation time. For instance, the computation of figure 6 took 3 minutes on a 1 GHz computer. Moreover, if the signal which had to be analysed was bidimensional, its Wigner distribution would be difficult to represent because we would have to process a 4-dimensions function. We must keep in mind that a multicomponent signal generates $\binom{N}{2}$ interference terms, which would make it harder to understand. The fractional Fourier transformation can solve all these drawbacks and is a powerful analysis tool to analyze linear chirped signals.

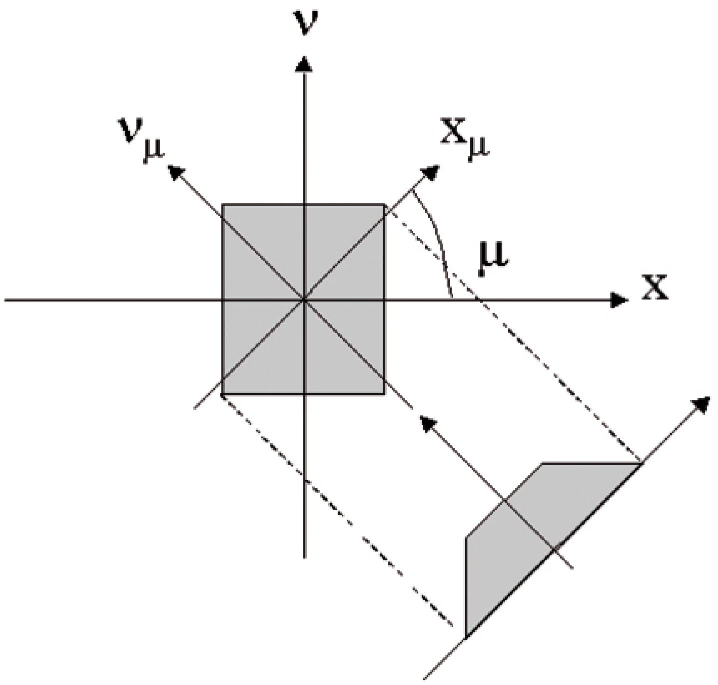


Figure 5 : Rotation direction of the axis in the Radon transform

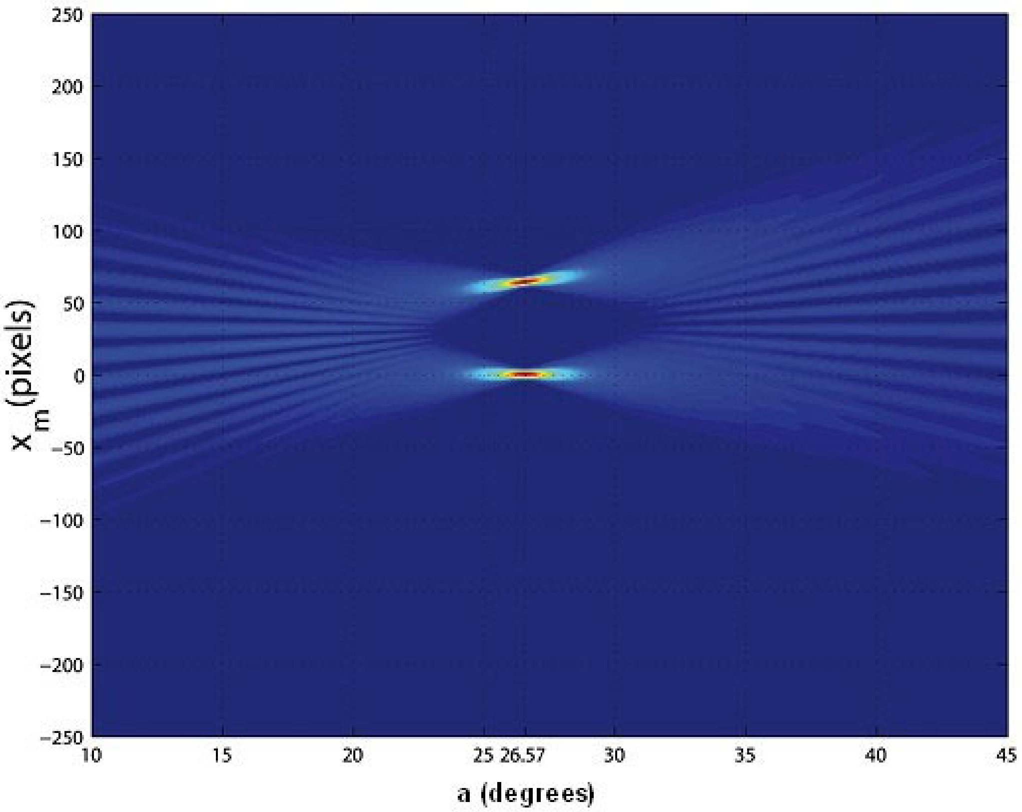


Figure 6 : Representation of the Radon transform of the energy density of the Wigner distribution of the signal $u_1(x) + u_2(x)$ *with angle range from 10 to 45 degrees*

2.4. Relation between FRFTs and Radon-Wigner distributions

Lohmann *et al*. gave a new definition of the Radon transformation (or projection theorem) by interpreting FRFTs through Wigner distributions [[Image rotation, Wigner rotation, and the fractional Fourier transform]]:

- The Wigner distribution of the fractional Fourier transform of a function $f(x)$ is the Wigner distribution of $f(x)$ by a clockwise rotation of the axes,
- The square module of the fractional Fourier transform of order α of a function $f(x)$ is the Radon-Wigner transform for the angle μ.

$$W_{fa}(x,v)=W_f(x\cos\alpha-v\sin\alpha\,,x\sin\alpha+v\cos\alpha)$$
$$(2.44)$$

or

$$W_f(x,v)=W_{fa}(x\cos\alpha+v\sin\alpha-x\sin\alpha+v\cos\alpha)$$
$$(2.45)$$

and

$$\mathscr{R}\,\mathscr{W}\,\mu[f](x_a)=|\mathscr{F}_\alpha[f(x)](x_a)|^2$$
$$(2.46)$$

with $f_a=\mathcal{F}_\alpha[f]$. To illustrate this, we will analyze the elementary function written $rect(x/d)$, defined by :

$$f(x)=\mathrm{rect}\left(\frac{x}{d}\right)=\begin{cases}1 & \text{si } |x|<d/2\\ 1/2 & \text{si } |x|=d/2\\ 0 & \text{sinon}\end{cases}$$
$$(2.47)$$

The study of this function is interesting because no linear chirp function appears in the fractional domain. This example uses the digital calculation of fractional Fourier transform which will be studied later in section D. The function $rect\left(\frac{x}{d}\right)$ and its associated Wigner distribution function are represented in figure 7. We can note the oscillations due to internal interferences explained in B.2 (example of a Wigner distribution function). We apply a fractional Fourier transformation of order $\alpha=\pi/4$ to the function $rect\left(\frac{x}{d}\right)$ and calculate its Wigner distribution. The results are presented in figure 8. In this case, the rotation of the Wigner distribution of $f(x)$directly depends on the value of the fractional order α. We can therefore determine the variation law of the instantaneous space frequency thanks to (2.34).

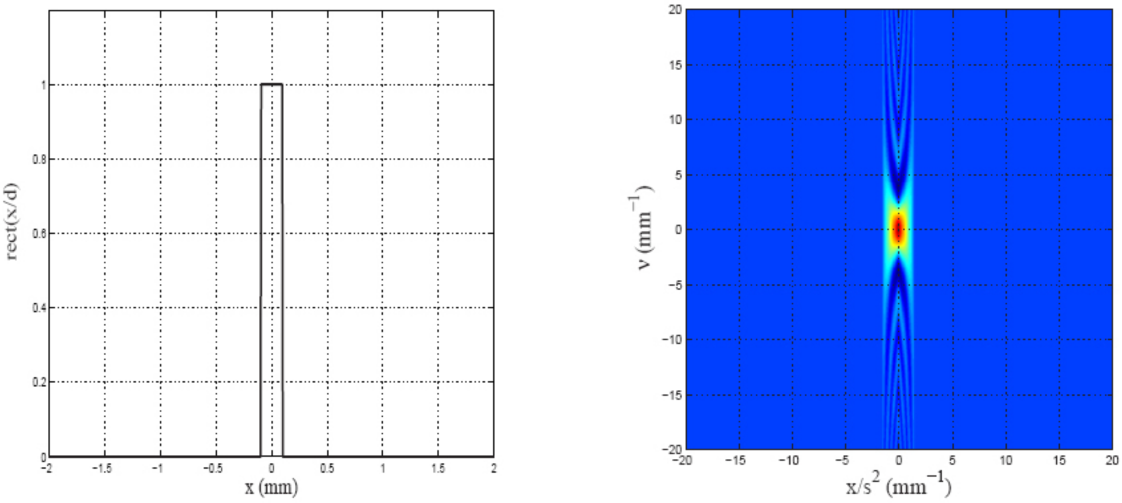


Figure 7 : Rectangular function with a $0.2\ mm$ *magnitude (*$d = 0.2$*) and its Wigner distribution function*

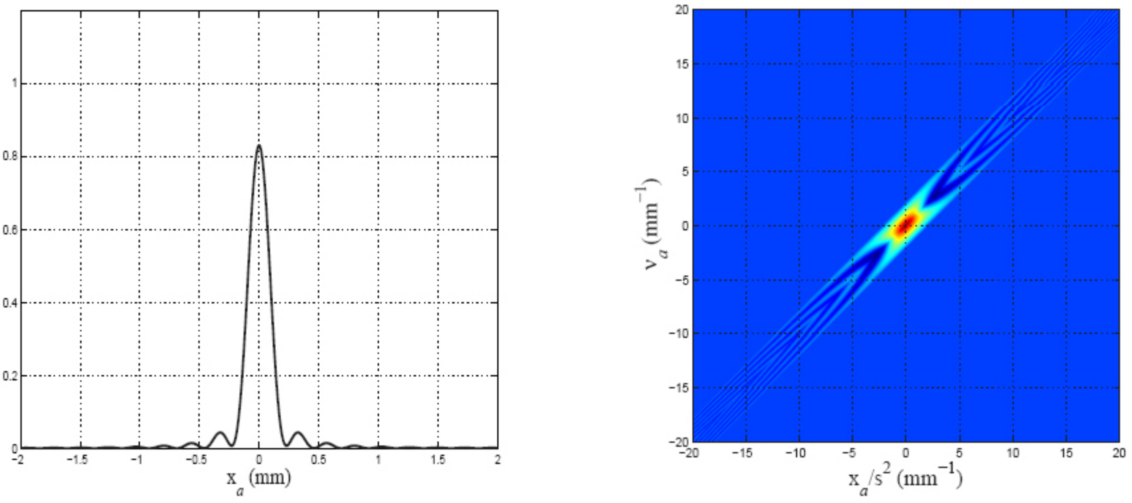


Figure 8 : Square module of the fractional Fourier transform of order $\alpha = \pi/4$ *with* $s^2 = 63.28 \times 10^{-3}\ mm^2$ *of the rectangular function with a* $0.2\ mm$ *magnitude (*$d = 0.2$*) and its Wigner distribution function*

It equals to :

$$v(x)=\frac{1}{2\pi}\frac{\partial}{\partial x}\left(\frac{\pi x^2}{\tan\alpha}\right)=x\cot\alpha \qquad (2.48)$$

The rotation angle corresponds to the angle $\alpha = \pi/4$ of the FRFT, as shown in figure 8. x_a is the variable associated with the fractional domain. Finally, we check if, with the fractional order $a = 1$, the rotation of the Wigner distribution is equal to $\pi/2$ (fig. 9) The line graphs of the square modules of the FRFTs strictly correspond to the projection of the associated Wigner distributions following an axis parallel to the abscissae x and x_a respectively.

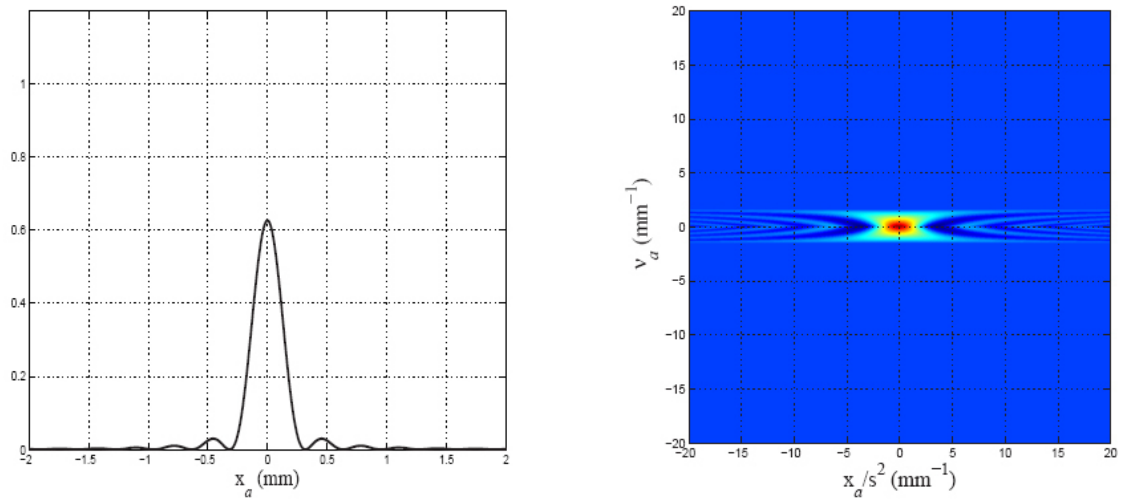


Figure 9 : Square module of the fractional Fourier transform of order $\alpha = 1$ *of the rectangular function with a* $0.2\ mm$ *magnitude (*$d = 0.2$*) and its Wigner distribution function*

3. Differential equation solving

3.1. Illustration and Resolving

The $FRFT$ used to solve a second order differential equation given by [[On Namias's Fractional Fourier Transforms]] will be illustrated in this section:

$$f''(x)+b^2x^2f(x)=0 \tag{2.44}$$

with $b > 0$. To simplify the expression, we will set $f_a = \mathrm{F}_\alpha f$. Now we will consider (2.44), and calculate its $FRFT$ such that:

$$\mathrm{F}_\alpha[f''(x)+b^2x^2f(x)]=\mathrm{F}_\alpha[f''(x)]+b^2\mathrm{F}_\alpha[x^2f(x)]=0 \tag{2.45}$$

Considering Eqs. (2.22) and (2.25), we have:

$$[-4\pi x_a^2\sin^2\alpha+i\pi\sin(2\alpha)]f_a+i2\pi\sin(2\alpha)x_a\frac{df_a}{dx_a}+\cos^2\alpha\frac{d^2f_a}{dx_a^2}+\frac{b^2\sin2\alpha}{4\pi}[i+2\pi x_a^2\cot\alpha]f_a+\frac{ib^2}{2\pi}x_a\sin(2\alpha)\frac{df_a}{dx_a}-\frac{b^2\sin^2\alpha}{4\pi^2}\frac{d^2f_a}{dx_a^2}=0 \tag{2.46}$$

now

$$[\cos^2\alpha-\frac{b^2\sin^2\alpha}{4\pi^2}]\frac{d^2f_a}{dx_a^2}+i2\pi\sin(2\alpha)x_a[1+\frac{b^2}{4\pi^2}]\frac{df_a}{dx_a}+[x^2(b^2\cos^2\alpha-4\pi2\sin^2\alpha)+i\pi\sin2\alpha(1+\frac{b^2}{4\pi^2})]f_a=0 \tag{2.47}$$

This new differential equation is established for any fractional domain α. We can also select the most interesting domain to solve our differential equation. Accordingly, reducing the differential equation order by suppressing the second order derived functions is desirable. So, we must have:

$$\cos^2\alpha - \frac{b^2\sin^2\alpha}{4\pi^2} = 0$$
$$(2.48)$$

Another way to write it would be:

$$\tan\alpha = \frac{2\pi}{b}$$
$$(2.49)$$

To solve our differential equation, we have to be in the fractional domain α such that:

$$\alpha = \arctan[\frac{2\pi}{b}]$$
$$(2.50)$$

In that case, the differential equation can be simplified to become a first order differential equation:

$$i\,2bx_a\frac{df_a}{dx_a} + [x_a^2(b^2-4\pi^2)+ib]f_a = 0$$
$$(2.51)$$

The solution of Eq. (2.51) is:

$$f_a(x_a) = \frac{K}{\sqrt{|x_a|}}\exp[i\frac{(b^2-4\pi^2)}{4b}x_a^2]$$
$$(2.52)$$

This is what we call the "spectral solution" of the differential equation in the fractional domain, defined by the fractional order $\alpha = arctan\left[\frac{2\pi}{b}\right]$. To obtain the final solution, we have to apply the inverse fractional Fourier transform.

$$f(x) = \mathsf{F}_{-\alpha}[f_a(x_a)](x) = C(-\alpha)e^{-i\frac{\pi x^2}{\tan\alpha}}\int_{-\infty}^{+\infty} f_a(x_a)e^{-i\frac{\pi x_a^2}{\tan\alpha}}e^{-i\frac{2\pi x_\alpha x}{\sin\alpha}}dx$$
$$(2.53)$$

As equation (2.52) is an even function, $\omega = |x_a|^{1/2}$, and we use Gradshteyn's relation such that:

$$\int_0^{+\infty}\cos[\beta\omega^2]\exp[i\gamma\omega^4]d\omega = \frac{\pi}{4}\left(\frac{|\beta|}{2}\right)^{1/2}\exp[\frac{i\pi}{8}-\frac{i\beta^2}{8\gamma}]J_{-1/4}[\frac{\beta^2}{8\gamma}]$$
$$(2.54)$$

where $J_{-1/4}$ is a Bessel function of order $-1/4$. The solution of this differential equation is:

$$f(x)=\frac{K\pi\sqrt{2b}}{2}C(-\alpha)\exp[i\frac{5\pi}{8}]\cdot|x|^{1/2}\cdot J_{-1/4}[\frac{bx^2}{2}] \quad (2.55)$$

As a conclusion, this solved example points out the fact that Fourier transformations are not helpful here. We can also see the advantage of introducing an arbitrary coefficient α to reduce the derivative order of the differential equation, and then solving this equation becomes possible.

4. Digital fractional Fourier Transforms

The digital implementation of the Fourier transformation by a discrete Fourier Transformation (DFT) is a real success. As far as $FRFT$ is concerned, there is no algorithm of FFT which combines the following advantages: calculation speed, property preservation and use flexibility. This is one of the reasons why fractional Fourier Transformation is not widely used yet in spectral analysis. However, there are two algorithm families. The first one, called fast algorithms, uses Fast Fourier Transformation (FFT) based on the definition of $FRFT$ given in (2.9). The second one uses matrix algorithms based on the definition given in (2.8).

4.1. From continuous to discrete

The digital calculation of conventional Fourier transformation is based on its discrete representation. Discrete Fourier transformation (DFT) is an appropriate sampling and a digital version of continuous Fourier transformation (CFT). The DFT and CFT of a function $f(x)$ are the same when it meets the following conditions:

- $f(x)$ is defined in a fixed set equal to the period of the signal,
- the sampling frequency is twice as large as the maximum frequency of $f(x)$,
- the sampling interval Δx is an integer multiple of the period of $f(x)$.

For non-periodic functions, $DFTs$ can be considered as equivalent to $CFTs$ if the function has a limited scope and if the gap in the highest frequencies has insignificant effects on the power spectrum. If we want to implement Fourier transforms by using $DFTs$, it is necessary to multiply the discrete function in the spatial domain by the sampling period δx:

$$F(\frac{k}{N\delta x})=\delta x\sum_{n=-N/2}^{N/2-1} f(n\delta x)\exp(-i2\pi\frac{nk}{N}) \quad (2.56)$$

The continuous form of $FRFT$ is as follows:

$$f_a(x_a)=C(\alpha)\exp(i\frac{\pi x_a^2}{\tan\alpha})\int_{-\infty}^{+\infty} f(x)\exp(i\frac{\pi x^2}{\tan\alpha})\exp(-i2\pi\frac{x_a x}{\sin\alpha})dx \quad (2.57)$$

where $f_a(x_a) = \mathsf{F}_\alpha[f(x)](x_a)$.

If we change the variable of Eq. (2.57) such that:

$$\xi = x_a / (\sin\alpha) \tag{2.58}$$

Eq. (2.57) can be written:

$$\hat{f}_a(\xi) = C(\alpha)\exp\left(i\pi\frac{1}{2}\sin(2\alpha)\xi^2\right)\int_{-\infty}^{+\infty} f(x)\exp\left(\frac{i\pi x^2}{\tan\alpha}\right)\exp(-i2\pi\xi x)\,dx \tag{2.59}$$

with $\hat{f}_a(\xi) = f(\xi sin\alpha)$. As we will study it later, this change of scale will be automatically taken into account within an optimized sampling. In accordance with Eq. (2.56), the discrete expression of equation (2.59) is therefore:

$$\hat{f}_a(k\delta\xi) = \delta x\, C(\alpha)\exp\left(i\pi\frac{1}{2}\sin(2\alpha)k^2\delta\xi^2\right)\sum_{n=-N/2}^{N/2-1} f(n\delta x)\exp\left(\frac{i\pi n^2\delta x^2}{\tan\alpha}\right)\exp(-i2\pi nk\,\delta\xi\,\delta x) \tag{2.60}$$

where $\delta\xi$ is the sampling period in the fractional domain x_a. If Δx is the compact set of the studied signal, the sampling period δx is $\Delta x/N$ and the sampling period in the fractional domain is $1/\Delta x$. Equation (2.60) finally becomes:

$$\hat{f}_a(k) = \frac{\Delta x}{N} C(\alpha)\exp\left(i\pi\frac{\sin(2\alpha)}{2\Delta x^2}k^2\right)\sum_{n=-N/2}^{N/2-1} f\left(n\frac{\Delta x}{N}\right)\exp\left(\frac{i\pi\Delta x^2}{\tan\alpha}\frac{n^2}{N^2}\right)\exp(-i2\pi nk) \tag{2.61}$$

This relation has not only the advantage of being easy to implement digitally but also saves calculation speed since only one FFT and two multiplications by the linear chirp functions are necessary. An optimized sampling still needs to be defined.

4.2. Sampling optimization

To optimize sampling, it is necessary to start from continuous $FRFTs$ to analyse the quadratic phases that appear in the integral. An optimized sampling should take into account the quadratic phases which have strong oscillations when they are far away from the origin. With this in mind, we must impose a fixed integration domain.

First, we will examine the term quadratic phase $\varphi(x) = \pi x^2 \cot\alpha$ in the integral. The instantaneous frequency, written f_i, is defined by:

$$f_i(x) = \frac{1}{2\pi}\frac{\partial\phi(x)}{\partial x} = x\cot\alpha \tag{2.62}$$

The instantaneous period of the quadratic phase equals to $1/f_i(x)$or $tan(\alpha/x)$. The smallest pseudo-period of the signal, written T_{min}, is obtained by taking half of the set of the function, i.e. $x = \Delta x/2$. In other words:

$$T_{mim}=\frac{1}{\mathrm{Max}(|f_i(x)|)}=\frac{1}{(|f_i(\delta x/2)|)}=\left|\frac{2\tan\alpha}{\Delta x}\right|$$
$$(2.63)$$

The sampling frequency, f_e, should be in accordance with Shannon's condition, that is to say it must be equal to or twice as great as the maximum instantaneous frequency of the signal: $f_e \geq 2f_{imax}$. Therefore:

$$\delta x \leq \left|\frac{\tan\alpha}{\Delta x}\right|$$
$$(2.64)$$

As far as the term quadratic phase outside the integral of the $FRFT$ is concerned, the process is strictly the same. Accordingly, the sampling period in the fractional domain must also conform to:

$$\delta x_a \leq \left|\frac{\tan\alpha}{\Delta x_a}\right|$$
$$(2.65)$$

Now we still have to define the relation between the sets Δx and Δx_a. This can be achieved through the uncertainty relation written according to the integration variables considered in Eq. (2.59):

$$\Delta x \Delta \xi = N$$
$$(2.66)$$

where N defines the number of sampling points. By using Eq. (2.57), uncertainty relation Eq. (2.66) becomes:

$$\Delta x \Delta x_a = N\sin\alpha$$
$$(2.67)$$

Eq. (2.67) shows that when a signal is chirped, then the uncertainty relation is broken: the product of the sets is no longer constant but depends on the chirp expressed by the fractional order α. It should also be noted that this important relation ensures an inverse $FRFT$. At this stage of the discussion, it is possible to establish a framework for the set Δx that we will not explain in detail (cf. [[Numerical calculation of fractional Fourier transforms with a single fast-Fourier-transform algorithm]]). Yet, it equals to:

$$\sqrt{(N/2)\sin 2\alpha} \leq \Delta x \leq \sqrt{N\tan\alpha}$$
$$(2.68)$$

In practice, to calculate a $FRFT$ of order α, the upper limit of the framework will be chosen: $\Delta x = tan\alpha$. In the light of previous results, we can note that the law of composition cannot be simply achieved if we do not choose the sets carefully. The figures (10) illustrate the $FRFT$ of a rectangular function with a $0.5\ mm$ magnitude ($d = 0.5$) and different fractional orders.

These results are consistent with the results presented by Marinho [[Numerical calculation of fractional Fourier transforms with a single fast-Fourier-transform algorithm]]. This enables us to consider various studies on digital holography and beam analysis.

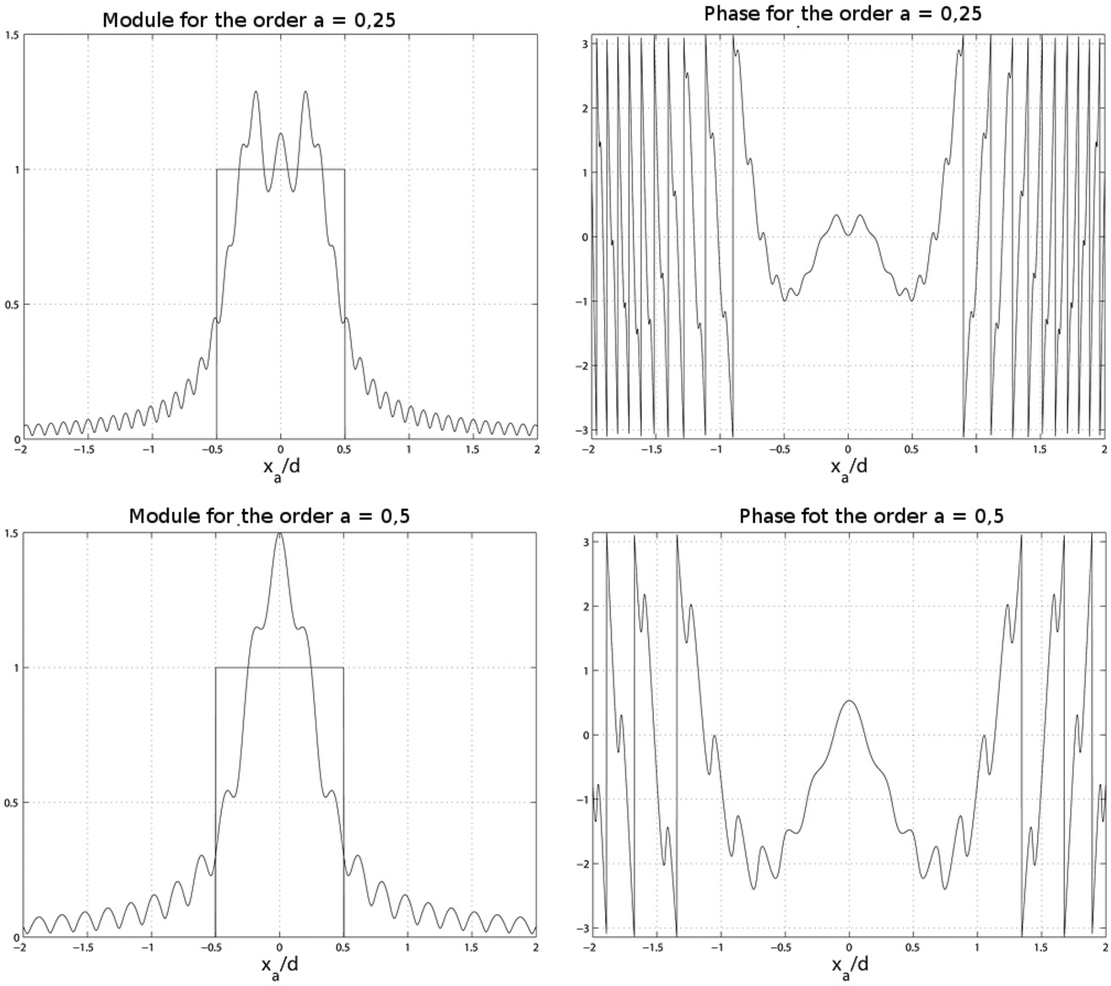


Figure 10 : Modules and phases of the $FRFT$ *of a rectangular function with a* $0.5\,mm$ *magnitude (*$d = 0.5$*)*

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