Estimating measurement uncertainties

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I. Présentation

Module :

Optical metrology

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Résumé :

Presentation of the approach to follow to evaluate the components of a measurement uncertainty. Application to measurement situations encountered in optics.

Mots-clés :

Systematic errors, random errors, standard uncertainty, expanded uncertainty, confidence interval, repeatability, reproducibility, correlation of measurements, probability laws

Pré-requis :

Descriptive statistics

Objectif(s) pédagogique(s) :

Be able to analyze a measurement system to determine the sources of uncertainty and evaluate the terms that compose it.

Plan du cours :

- Introduction
- Preliminary remarks
- Evaluation of standard uncertainty
- Composite standard uncertainty
- Determination of expanded uncertainty
- Presentation of measurement results
- Summary of the uncertainty assessment procedure
- Conclusion

Conception & production :

Le Mans Université

Licence :

II.Course

In the optical field as in other experimental sciences, there is no such thing as an *« exact measurement »*. Measurements will always be subject to error, to which extent will depend on the method chosen, the operating procedure, the quality of the measuring instruments or the skills of the operator.

Evaluating the uncertainty of a measurement requires a complex method which constitutes a branch of sciences called "metrology". In order to give this evaluation with a large and universally recognized consensus, a guide has been set up for the expression of measurement uncertainties. Its French version is the NF ENV 13005 norm dated August 1999 [1 [Guide pour l'expression de l'incertitude de mesure, 1999]]. As for the normative vocabulary, the NF X07-001 norm of December 1994 [2 [Vocabulaire international des termes fondamentaux et généraux de métrologie, 1994]] contents all the International vocabulary of metrology (VIM).

Basic vocabulary

The formats and general terms of table 1 will be used throughout the lesson.

Écriture	Signification
Y	Measurand, quantity intended to be measured
y	Measurement of the quantity Y
$u(Y)$	Standard uncertainty
$U(Y)$	Expanded uncertainty
$\frac{U(Y)}{y}$	Relative uncertainty

Note :

- Do not confuse Y and y : Y designates the quantity which is the subject of a measurement, while y designates the numerical result of this measurement.
- La notation u et U provient de l'anglais «uncertainty»

Types of measurements

The measurement of a quantity Y can be obtained :

- either directly; for instance, when a distance $X$is measured with a ruler,
- or indirectly; for instance, when a displacement L is measured by an interferometric method.

In the first case, the functional relation is simple, of type $Y = X$ or $Y = \bar{X}$, if N replicate measurements of the distance $X$are performed and that we consider their average value $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$.

In the second case, the displacement L is such as $L = p\frac{\lambda_{VACUUM}}{n(T,P,H)}$ where p is an integer , λ_{VACUUM} is the wavelength in vacuum of the luminous source used in the interferometer and $n(T,P,H)$ is the index of the medium (air for example) where light rays propagate; this index is a function of the ambient medium's temperature T, of its pressure P and of its hygrometry degree H.

In general, we will consider $Y = f(X_1, X_2, \ldots)$ where $X_1, X_2, \ldots, X_j$ will be input quantities that will generally be directly measured.

1. Introductory notes

When reporting the measurement result of a physical quantity, one must give a quantitative indication of the quality of such measurement result, so that those who will use it can estimate its reliability. If such an indication is missing, the result will be impossible to compare these measurement results, either to one another, or to reference quantity values set by a specification or norm.

The concept of uncertainty as a quantifiable attribute of a measurement result's quality is rather new in the history of measurement, although error and error analysis have long been part of metrology. It is actually considered that once all the known or suspected components of an error have been evaluated, and once the appropriate corrections have been made, there will still remain an uncertainty regarding the validity of the expressed result, that is to say a doubt on the extent to which the measurement result correctly accounts for the measured quantity's value.

The official definition given by the VIM (§ 3,9) for the term « measurement uncertainty »is thus the following :

From a practical point of view, the uncertainty of a measurement will be expressed under the form of a standard deviation (in a statistical sense), and will be referred to as standard uncertainty $u(Y)$.

2. Evaluation of standard uncertainty

An estimate of the measurand Y, noted y, is obtained from the equation $Y = f(X_1, X_2, ...)$ called **mathematical model of measurement** , by using the estimates $x_1, x_2, ..., x_j$of the input quantities $X_1, X_2, ..., X_j$. The standard deviation associated with the output estimate or the measurement result y, called **combined standard uncertainty** and noted $u_c(y)$, determined from the estimated standard deviation associated with each input estimate $x_i$called **standard uncertainty** and noted $u(x_i)$. Each input estimate x_i and its associated standard uncertainty $u(x_i)$ are obtained from a distribution law of the possible values of the input quantity X_i. This probability law can be based on a series of replicate observations $X_{i,k}$of the several X_i,in which case we will refer to a **évaluation de Type A** of the components of the standard uncertainty. It can also be an a priori law, and will therefore correspond to a **Type B evaluation.** In both cases, the laws that are used depend on our level of knowledge of the measurement mean.

2.1. Type A evaluation

This is the case when the operator performs a series of replicate measurements in repeatability conditions of measurement (cf. VIM §3.6).

The arithmetic mean $\bar{X}_i$ obtained from the equation $\bar{X}_i = \frac{1}{N} \sum_{k=1}^{N} X_{i,k}$ is used as the best estimate of the input quantity X_i. The values of the individual observations $x_{i,k}$iffer due to the random variations of the influence quantities. The variability of the observed values $x_{i,k}$ or, more specifically, their dispersion around their mean value $\bar{x}_i$ is called **experimental standard deviation** and is noted :

$$s(x_i) = \sqrt{\frac{1}{N-1} \sum_{k=1}^{N} (x_{i,k} - \bar{x})^2}$$

We can deduce from it the **experimental standard deviation of the average** $s(\bar{x}_i)$ such as $s(\bar{x}_i) = \dfrac{s(x_i)}{\sqrt{N}}$, that is to say:

$$s(\bar{x}_i)=\sqrt{\frac{1}{N(N-1)}\sum_{k=1}^{N}(x_{i,k}-\bar{x})^2}$$

In practice, the experimental standard deviation of the average is called **uncertainty of repeatability**.

2.2. Type B evaluation

If a measurement laboratory was to be granted with unlimited resources and time, it could carry out an exhaustive statistical study of every imaginable cause of uncertainty, using for instance different types of devices made by different manufacturers, with different measurement methods, different operating procedures and different approximations in the different measurement theoretical models.

It would then be possible to estimate the uncertainties associated with all these causes through the statistical analysis of the series of observations, and to characterize the uncertainty due to each cause with a statistically evaluated standard deviation. Eventually, all components of the uncertainty would result from Type A evaluations.

But since such a study is financially impossible, numerous components of the uncertainty have to be evaluated through all the other feasible means. The information that will be analysed can include:

- results from previous measurements,
- general or empirical knowledge of the behavior of the instruments used,
- specifications of the manufacturer,
- calibration certificates,
- the uncertainty attributed to reference quantity values mentioned in studies, textbooks or norms.

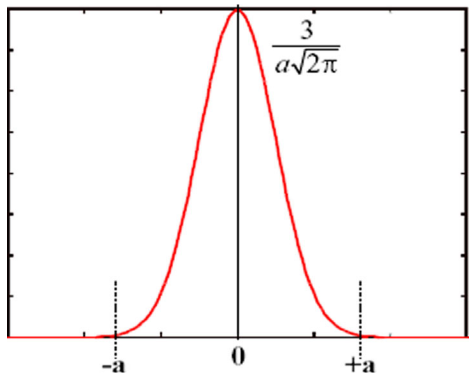
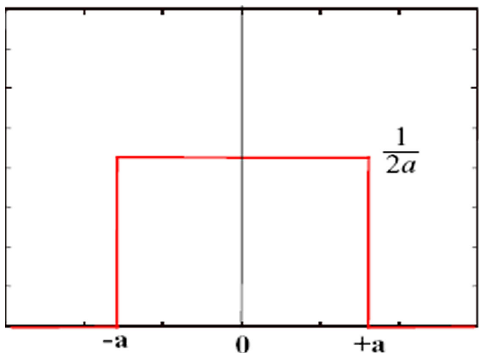
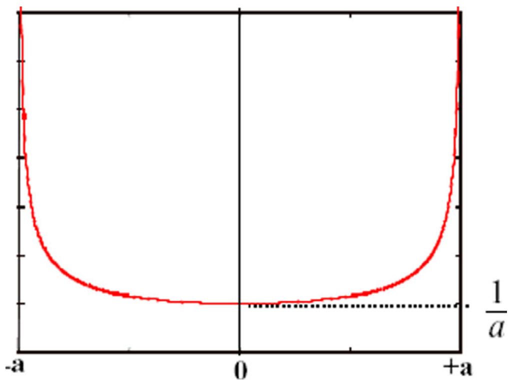
Thus for an estimate x_i of a quantity X_i which has not been obtained from replicate observations, the estimated variance $u^2(x_i)$ or the standard uncertainty $u(x_i)$ est évaluée par un is evaluated through a scientific judgement based on all the available information regarding the possible variability of X_i. The standard uncertainty evaluated by this mean is called **Type B standard uncertainty.**

In practice, a balance of all the errors must be carried out. These errors are divided into:

- Systematic measurement errors (cf. VIM §3.14) such as the parallax error when reading a needle dial, the zero adjustment of a device, method errors, components ageing, etc.
- Random error of measurement (cf. VIM §3.13) such as reading errors or errors due to the device itself, or to exterior conditions (temperature, thermal expansion, atmospheric pressure, humidity, etc.).

a) A priori probability laws

In order to express Type B uncertainty under the form of a standard deviation, probability laws must be used. Table 2 presents the most commonly used probability laws, referring here to a distribution of values of a random variable of mean $\mu = 0$ and of range $[-a\,;+a] = 2a$.

Loi	Représentation graphique	Écart-type
Normal or Gaussian law $a = 3\,\sigma$		$\frac{a}{3}$
Uniform or rectangular law		$\frac{a}{\sqrt{3}}$
Derivative of arcsine law		$\frac{a}{\sqrt{2}}$

Generally, if the manufacturer provides the standard uncertainty, it is used directly.

If very little information is available on an input quantity and its supposed variation interval comes under the form:

- $\Delta x = \pm a$ while standard uncertainty is : $u(x) = \dfrac{a}{\sqrt{3}}$.
- $\Delta x = q$ while standard uncertainty is : $u(x) = \dfrac{q}{2\sqrt{3}}$.

considering a uniform law in the variation interval of the quantity.

b) Examples of Type B uncertainties

- **Resolution of a measuring instrument**

  The graduation of a measuring instrument or a digital display device is a source of uncertainty. If the resolution of the indicating device is δ_x, Ithe value of the input signal producing a given indication X can lie with equal probability anywhere in the interval

$\left[X - \frac{\delta x}{2} ; X + \frac{\delta x}{2}\right]$, the input signal is then described by a rectangular probability law of width δ_x and of standard deviation $u_{res}(x) = \frac{\delta x}{2\sqrt{3}}$ called **resolution uncertainty.**

- **Class of an instrument**

  The maximum permissible measurement error (cf. VIM §5.21) gives the extreme variation limits of the indication obtained from a measuring instrument, the class of which is defined by the interval $[-a ; +a]$. The associated standard uncertainty is then $u_{class}(x) = \frac{a}{\sqrt{3}}$.

- **Hysteresis**

  The different indications of a measuring instrument can be a fixed quantity, depending on the successive readings being made in ascending or decreasing order of values. Most of the time the direction of the hysteresis cannot be observed. If the width of the range of the possible readings due to this cause is δ_x, the standard uncertainty due to this hysteresis is $u_{hyst}(x) = \frac{\delta x}{2\sqrt{3}}$.

- **Temperature variations**

  One of the main influences on the quantity of a measuring system is the temperature of the means of measurement's environment (room, air-conditioned compound, case, etc.). Insofar as the temperature would vary between two extremes in a quasi-sinusoidal way, the law of probability associated with this influence quantity is the derivative of the arcsine function. If the temperature variations are such as $\Delta T = \pm b$, then the standard uncertainty due to temperature variations is $u_{temp}(T) = \frac{b}{\sqrt{2}}$.

3. Combined standard uncertainty

3.1. Direct measurement of quantity Y

Most of the time Type A and Type B uncertainties must be combined so that:

$$u(y)=\sqrt{u_A^2+u_B^2}$$

that is to say

$$u(y)=\sqrt{u_{rép}^2+u^2(instrument)+u^2(others)}$$

with

$u^2_{rép}$:	Repeatability uncertainty
u^2(instrument):	(Type B) standard uncertainty of the measuring instrument, taking into account the contributions described in § IV.2.b. ii, for instance : $u^2(\text{instrument}) = u^2_{rés}(x) + u^2_{hyst}(x)$
u^2(others):	Other (Type B) standard uncertainties than the ones associated with the measuring instrument (previous measurement results, experimenter, etc.)

a) Case of a single measurement

Since only one measurement is performed, $u_{rep} = 0$ hence :

$$u(y)=\sqrt{u^2(instrument)+u^2(others)}$$

Exemple

An electric current is measured just once with an ammeter with a standard uncertainty of 2.9 mA. Previous measurement results have given a standard uncertainty of 5.2 mA5.2 mA

The standard uncertainty on the value of the measured intensity I is :

$$u(I)=\sqrt{u^2(ammeter)+u^2(others)}=[(2,9)^2+(5,2)^2]^{1/2}=5,9 \Rightarrow u(I)=5,9\,mA$$

b) Case of replicate measurements

In this case the repeatability uncertainty is evaluated from the N replicate measurements.

Exemple

The same measurement of current is performed 9 times with an ammeter with a standard uncertainty of 2.9 mA2.9 mA.

$$I(A)=2,16;2,12;2,15;2,15;2,17;2,18;2,16;2,15;2,14$$

No other information is given or available.

The standard uncertainty on the value of the measured intensity I is :

$$u(I)=\sqrt{u^2_{rép}+u^2(ammeter)}=[(5,77)^2+(2,9)^2]^{1/2}=6,46 \Rightarrow u(I)=6,5\,mA$$

3.2. Indirect measurement of quantity Y

For a measurand Y function of several input quantities $X_i$ollowing the mathematical model of the measurement $Y = f(X_1, X_2, \ldots)$, the standard uncertainty of y is obtained from the combination of the standard uncertainties of the input estimates *kx_{i}*. This **combined standard uncertainty** of the estimate y is noted$u_c(y)$.

Insofar as the function f does not present an important nonlinearity, it is expanded around the mathematical expectations $E(x_i) = \mu_i$ of the input quantities x_i. The expansion of first-order Taylor series gives, for small variations of y around μ_y :

$$y-\mu_y=\sum_{i=1}^{p}\frac{\partial f}{\partial x_i}(x_i-\mu_i)$$

The square of the difference $y-\mu_y$ is thus given by:

$$(y-\mu_y)^2=[\sum_{i=1}^{p}\frac{\partial f}{\partial x_i}(x_i-\mu_i)]^2$$

which can be written under the form of

$$(y-\mu_y)^2=\sum_{i=1}^{p}(\frac{\partial f}{\partial x_i})^2(x_i-\mu_i)^2+2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(x_i-\mu_i)(x_j-\mu_j)$$

The mathematical expectation of $(y-\mu_y)^2$ is the variance of y, that is to say $E\left[\left(y-\mu_y\right)^2\right]=\sigma_y^2=u_c^2(y)$ hence, finally:

$$u_c^2(y)=\sum_{i=1}^{p}(\frac{\partial f}{\partial x_i})^2u^2(x_i)+2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i)u(x_j)r_{ij}$$

with
$u(x_i)$ = tandard uncertainty on x_i
r_{ij}= correlation coefficient of x_i and x_j

This equation is called **propagation of uncertainties law**. It shows how the uncertainties $u(x_i)$ of the input quantities x_i combine to give the uncertainty $u_c(y)$ of the output quantity y.

The partial derivatives $\frac{\partial f}{\partial x_i}$ are called **sensitivity coefficients**. They describe how the output estimate y en fonction des variations dans les valeurs des estimations d'entrée $x_1, x_2, \ldots, x_n$.

The correlation coefficient r_{ij} is such as $r_{ij}=\frac{u(x_i,x_j)}{u(x_i)u(x_j)}$ where $u(x_i,x_j)=E\left[(x_i-\mu_i)\left(x_j-\mu_j\right)\right]$ is the covariance of x_i and ${}_j x_j$

a) Non-correlated input quantities

When all the input quantities $X_1, X_2, \ldots, X_n$ are independent, that is to say when the covariances $u(x_i, x_j)$ and the correlation coefficients r_{ij} are null, then the combined standard uncertainty is such as:

$$u_c^2(y)=\sum_{i=1}^{p}(\frac{\partial f}{\partial x_i})^2u^2(x_i)$$

i Cas où Y est une somme ou une différence

$$Y=A\,x_1\pm B\,x_2 \text{ then } u_c(Y)=\sqrt{[A\,u(x_1)]^2+[B\,u(x_2)]^2}$$

Exemple

Let *distance* L be the quantity depending on the measured quantities *position* x_1 et *position* x_2 such as $L = x_2 - x_1$. Then $u(L) = \sqrt{u^2(x_1) + u^2(x_2)}$. If $u(x_1) = u(x_2) = u$ then $u(L) = \sqrt{2}u$.

ii Case where Y is a product or a quotient

$$Y = A\frac{x_1^a x_2^b}{x_3^c} \text{ then } \frac{u_c(Y)}{Y} = \sqrt{[a\frac{u(x_1)}{x_1}]^2 + [b\frac{u(x_2)}{x_2}]^2 + [c\frac{u(x_3)}{x_3}]^2}$$

Exemple

Let *illumination* E be the quantity depending on the known or measured quantities *intensity* I_L and *distance* D. Bouguer's law gives $E = \frac{I_L}{D^2}$ that is to say :

$$\frac{u(E)}{E} = \sqrt{[\frac{u(I_L)}{I_L}]^2 + [2\frac{u(D)}{D}]^2}$$

b) Correlated input quantities

If the same physical standard scale, the same measuring instrument, the same reference data or the same measuring method is used in the estimate of the input quantities values, then a correlation will exist between input quantities.

In general, if two input quantities X_1 and X_2 estimated by x_1 and x_2 depend on a set of non-correlated variables $Q_1, Q_2, \ldots, Q_L$ such as $X_1 = F(Q_1, Q_2, \ldots, Q_L)$ and, $X_2 = G(Q_1, Q_2, \ldots, Q_L)$, the covariance associated with x_1 and x_2 is given by :

$$u(x_1, x_2) = \sum_{l=1}^{L} \frac{\partial F}{\partial q_l} \frac{\partial G}{\partial q_l} u^2(q_l)$$

with $u^2(q_1)$ being the variance associated with the estimate q_1 of Q_1.

The correlation coefficient estimated $r(x_1, x_2)$ is determined from the expression

$$r(x_1, x_2) = \frac{u(x_1, x_2)}{u(x_1)\, u(x_2)}$$

with $u^2(x_1) = \sum_{l=1}^{L} \left(\frac{\partial F}{\partial q_l}\right)^2 u^2(q_l)$ and a similar expression for $u^2(x_2)$.

Exemple

Ten electric resistors, each of a nominal value $R_i = 1000\ \Omega$ re calibrated in comparison with the same standard resistor $R_S = 1000\ \Omega$ characterized by a standard uncertainty $u(R_S) = 100\ \text{m}\Omega$.

The calibration of each resistor can be represented by the mathematical model $R_i = \alpha_i R_S$, with the standard uncertainty $u(\alpha_i)$ on the measured ratio α_i obtained from replicate observations. We suppose that $\alpha_i \simeq 1$ for each resistor and that $u(\alpha_i)$ is almost identical for each calibration so that $u(\alpha_i) \simeq u(\alpha)$.

The ten resistors R_i all depend on the same variable R_S because of their calibration. The covariance associated with R_i and R_j is

$$u(R_i, R_j) = \frac{\partial R_i}{\partial R_S} \frac{\partial R_j}{\partial R_S} u^2(R_S) \simeq u^2(R_S)$$

and

$$u^2(R_i) = \left(\frac{\partial R_i}{\partial \alpha_i}\right)^2 u^2(\alpha) + \left(\frac{\partial R_i}{\partial R_S}\right)^2 u^2(R_S)$$

that is to say

$$u^2(R_i) = R_S^2 u^2(\alpha) + u^2(R_S)$$

Therefore, the correlation coefficient of any two resistors $(i \neq j)$ is

$$r(R_i, R_j) = r_{ij} = \left[1 + \left(\frac{u(\alpha)}{u(R_S)/R_S}\right)^2\right]^{-1}$$

The estimated values of the resistors are thus correlated, with a correlation degree which depends on the ratio between the uncertainty of the comparison $u(\alpha)$ and the uncertainty of the reference standard $u(R_S)$. When the uncertainty of the comparison is negligible in relation to the uncertainty of the standard, the correlation coefficients r_{ij}re equal to $+1$ and the uncertainty of each calibrated resistor $u(R_i)$ is the same as the one of the standard.

Just as the estimated variance associated with an input quantity includes a statistical component (of Type A) and an evaluated component (of Type B), the estimated covariance associated with two input quantities can also include two contributions of Type A and Type B. Therefore the estimated covariance of two quantities X and Z , which are themselves estimated by the averages $\bar{X} \wedge \bar{Z}$ determined from N independent pairs (x_1, z_i) of replicate simultaneous observations, is given by $u(x,z) = s(\bar{x}, \bar{z})$ with

$$s(\bar{x}, \bar{z}) = \frac{1}{N(N-1)} \sum_{i=1}^{N} (x_i - \bar{x})(z_i - \bar{z})$$

The estimated correlation coefficient between the two quantities X and$Z$ ist then

$$r(\bar{x}, \bar{z}) = \frac{s(\bar{x}, \bar{z})}{s(\bar{x})\, s(\bar{z})}$$

Exemple

If the frequency of an oscillator with no temperature compensation is an input quantity and if the ambient temperature is also an input quantity and these two quantities are simultaneously observed, then there can be a significant correlation that the calculated covariance of the oscillator's frequency and the ambient temperature will highlight.

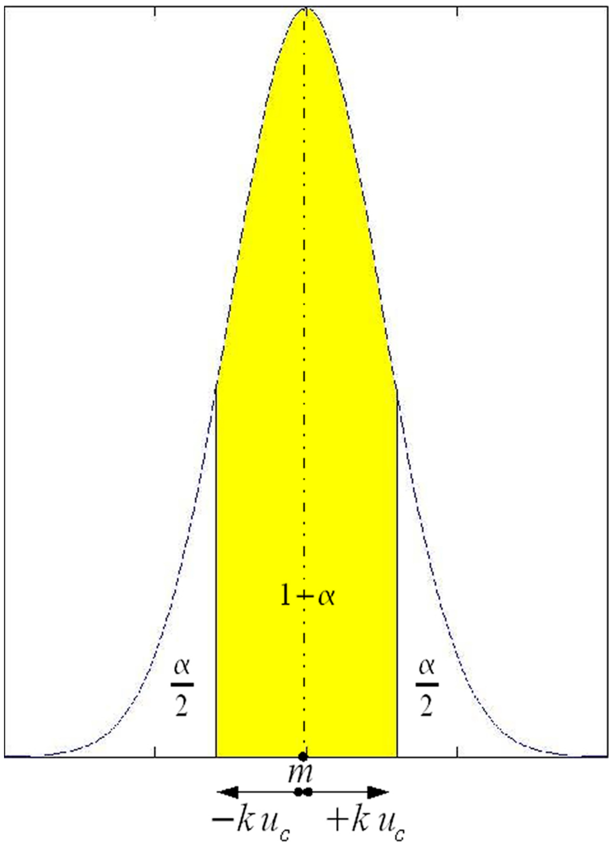
4. Determination of the expanded uncertainty

Although the combined standard uncertainty $u_c(y)$ can be sufficient to express the uncertainty of a measurement result, it is almost always necessary to provide a measurement of the uncertainty which will define, around the measurement result, an interval in which we will be able to find a large part of the distribution of values that could reasonably be attributed to the measurand.

In general, the measurement results are distributed around an average value m which is the measurand's most probable estimate. We can define an interval $[m - U\ ; m + U]$ such as the probability $P = 1 - \alpha$ where α is called confidence threshold $(0 \leq \alpha \leq 1)$ and $1 - \alpha$ confidence level. We have therefore defined a *confidence interval* of width $\pm U$ around the measurand's average value. U is obtained by multiplying the combined standard uncertainty $u_c(y)$ by a **coverage factor** *k* such as $U = ku_c(y)$ is called **expanded uncertainty.**

4.1. Choosing a coverage factor

If the measurement results are distributed according to a normal law around the average value m (cf. figure 1), table 3 shows the respective values of the coverage factor k and the confidence level $1 - \alpha$ sont rassemblées dans le tableau 3.



Coverage factor k	Confidence level 1 - α
1	0,68
1,96	0,95
2	0,95
2,58	0,99
3	1

These values are only strictly valid when the number N of replications of measurements is high (typically $N \geq 30$). This is rarely the case in practice, and we must therefore use *Student's t-distribution* followed by the variable $t = (\bar{x} - \mu_x)/s(\bar{x})$ where $\bar{x}$ is the arithmetic mean of N independent observations x_k of x and $s(\bar{x}) = s(x)/\sqrt{N}$ is the experimental standard deviation of the average $\bar{x}$. Note that Student's t-distribution is only valid if the random variable x follows a normal law of mathematical expectation μ_x and of standard deviation σ.

As a consequence, if the measurand Y is a single quantity X following a normal law, such as $Y = X$ and if X is estimated by the arithmetic mean X of N independent replicate observations X_k of X, with an experimental standard deviation of the average $s(\bar{X})$, then $t = (\bar{x} - \mu_x)/s(\bar{x}) = (\bar{X} - \mu_X)/s(\bar{X}) = (y - Y)/u_c(y)$ is distributed according to Student's t-distribution with

$$\text{Probability}\,[-t_{1-\alpha}(\nu) \leqslant t \leqslant t_{1-\alpha}(\nu)] = 1 - \alpha$$

that is to say

$$\text{Probability}\,[-t_{1-\alpha}(\nu) \leqslant (y - Y)/u_c(y) \leqslant t_{1-\alpha}(\nu)] = 1 - \alpha$$

or

$$\text{Probability}\,[y - t_{1-\alpha}(\nu)\,u_c(y) \leqslant Y \leqslant y + t_{1-\alpha}(\nu)\,u_c(y)] = 1 - \alpha$$

In these expressions, $t_{1-\alpha}(\nu)$ is the value of t for a given value of the parameter ν (number of degrees of freedom) such as the interval $[-t_{1-\alpha}(\nu)u_c(t)\,; t_{1-\alpha}(\nu)u_c(y)]$ is associated with a confidence level $1 - \alpha$. In other words, the expanded uncertainty is

$$U = t_{1-\alpha}(\nu)\,u_c(y)$$

4.2. Number of degrees of freedom

The number of degrees of freedom ν is equal to $N - 1$ in the case of the direct measurement of a quantity estimated by the arithmetic mean of N independent observations. If these N observations are used to determine the slope a observations are used to determine the slope b of a straight line by the least-squares method (case of a calibration straight line such as $Y = aX + b$), the number of degrees of freedom respectively associated with the standard uncertainties $u(a)$ and $u(b)$ is $\nu = N - 2$. For an adjustment by the least-squares method of p parameters for N data, the number of degrees of freedom of the standard uncertainty of each parameter is $\nu = N - p$. Table 4 sums up the different enumerated cases.

Degrees of freedom ν	Situation de mesure
N – 1	N Replicate measurements
N – 2	N Replicate measurements + calibration straight line Y=a.X+b
N – p	N Replicate measurements + adjustment to p parameters

Table 5 shows a selection of values of $t_{1-\alpha}(\nu)$ for different values of the confidence level $1 - \alpha$ and of the number of degrees of freedom ν.

Number of degrees of freedom ν	Level of confidence $1-\alpha$ in %					
	68,27	90	95	95,45	99	99,73
	Value of $t_{1-\alpha}(\nu)$					
1	1,84	6,31	12,71	13,97	63,66	235,8
2	1,32	2,92	4,30	4,53	9,92	19,21
3	1,20	2,35	3,18	3,31	5,84	9,22
4	1,14	2,13	2,78	2,87	4,60	6,62
5	1,11	2,02	2,57	2,65	4,03	5,51
6	1,09	1,94	2,45	2,52	3,71	4,90
7	1,08	1,89	2,36	2,43	3,50	4,53
8	1,07	1,86	2,31	2,37	3,36	4,28
9	1,06	1,83	2,26	2,32	3,25	4,09
10	1,05	1,81	2,23	2,28	3,17	3,96
15	1,03	1,75	2,13	2,18	2,95	3,59
20	1,03	1,72	2,09	2,13	2,85	3,42
25	1,02	1,71	2,06	2,11	2,79	3,33
30	1,02	1,70	2,04	2,09	2,75	3,27
35	1,01	1,70	2,03	2,07	2,72	3,23
50	1,01	1,68	2,01	2,05	2,68	3,16
100	1,01	1,66	1,984	2,025	2,626	3,077
∞	1,00	1,645	1,96	2	2,576	3,00

When $\nu \to \infty$, Student's t-distribution tends towards normal law and $t_{1-\alpha}(\nu) \simeq k\sqrt{1+\frac{2}{\nu}}$ where k is the necessary coverage factor to obtain a confidence interval of level $1-\alpha$ for a normally distributed variable. Thus in table 5 the value of $t_{1-\alpha}(\infty)$ for a given level of confidence $1-\alpha$ is equal to the value of k for the same value of $1-\alpha$ in table 3. From the expression of $t_{1-\alpha}(\nu)$, we can also evaluate the number of degrees of freedom ν_L for which $t_{1-\alpha}(\nu_L) = 1,1 \times k$ enabling us to evaluate the number of replicate measurements $N_L = \nu_L + 1$ beyond which Student' s t-distribution is less than 10% from a normal law, that is to say :

$$1,1 \times k = k\sqrt{1+\frac{2}{\nu_L}}$$

hence

$$\nu_L = 9,53$$

that is to say

$$N_L \geqslant 10$$

Therefore, at least 10 replicate measurements are needed to approach with a 10% accuracy a normal law describing the distribution of the values of the measured quantity around its average value.

Student's t-distribution does not generally describe the law of the variable $t = (y - Y)/u_c^2(y)$ if $u_c^2(y)$ is the sum of several components of variance estimated $u_i^2(y) = \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) = c_i^2 u^2(x_i)$ even if each x_i is the estimate of a normally distributed input quality X_i . However it is still possible to use Student's t-distribution with an effective number of degrees of freedom ν_{eff} obtained by the Welch-Satterthwaite formula [3a [Publication revue, 1938]][3b [Publication revue, 1947]][4 [Publication revue, 1946]] .

$$\nu_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{p} \frac{u_i^4(y)}{\nu_i}}$$

where $\nu_i$is the number of degrees of freedom of each component of the combined standard uncertainty $u_c(y)$ for $u_c^2(y) = \sum_{i=1}^{p} u_i^2(y) = \sum_{i=1}^{p} c_i^2 u^2(x_i)$. For a component obtained from a Type A evaluation, $\nu_i$is associated with the number of independent replicate observations of the corresponding input quantity and with the number of parameters determined from these observations (cf. table 4). For a component obtained from a Type B evaluation, $\nu_i$is evaluated from the reliability that can be given to the value of this component following the expression

$$\nu_i \simeq \frac{1}{2}\left[\frac{\Delta u(x_i)}{u(x_i)}\right]^{-2}$$

where $\frac{\Delta u(x_i)}{u(x_i)}$ is the relative uncertainty of $u(x_i)$. It is a subjective quantity which value is obtained from a scientific judgement based on all the available information. Note that if $u(x_i)$ can be considered as exactly known then $\nu_i \to \infty$.

5. Presentation of measurement results

In general, a measurement result consists of four elements:

- a numerical value which corresponds to the measurand's value and may be corrected if an accuracy error has been noticed,
- an expanded uncertainty associated with a confidence interval,
- a measurement unit guaranteeing the traceability with the International System of Units (S.I.),
- a coverage factor of the standard uncertainty.

The expression of the measurement result is then under the following form:

$$Y = y \pm U \; S.I.\,units\,(k = 2 \text{ or value of factor } t)$$

Besides, the following question must be asked when expressing the measurement result:

«Has enough information been provided in order to be able to update the result later if new information or data were to be made available? »

An excess of information is always better than a lack of it. For example, we must make sure that we :

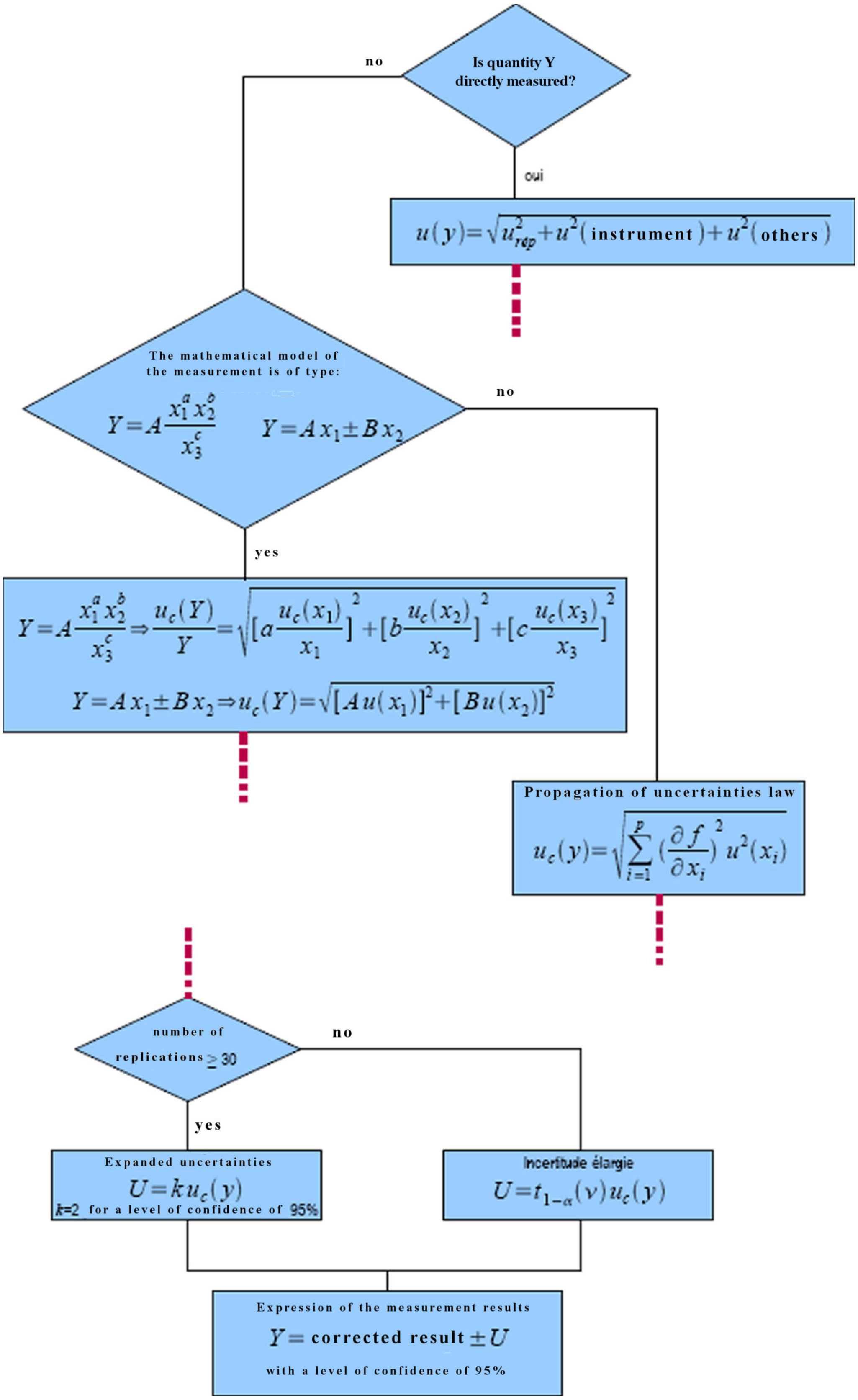
- give a clear description of the methods used to calculate the measurement result and its uncertainty on the basis of the experimental information and the input data,

- list all the components of the uncertainty and provide documentation describing thoroughly how they have been evaluated,
- present the analysis of the result in such a way that each major step can be easily retraced and that the provided result might be calculated again if necessary,
- give all the corrections and the constants used for the analysis, along with their sources.

The numerical values of the estimate y and of its standard uncertainty $u_c(y)$ or its expanded uncertainty U must not be given with an excessive number of figures. It is usually enough to give $u_c(y)$ et U and all the standard uncertainties $u(x_i)$ with two significant figures. However, it might be necessary to mention more figures for the different $u(x_i)$ in order to avoid the propagation of rounding errors in intermediate calculations. The table below sums up the general principles that must be respected when presenting numerical values.

Rules to be followed	**Let *E* be the illumination displayed by a luxmeter** *E* = 100,23465 *lux* **with** *u(E)*=0,104 *lux* **or** *U(E)*=0,208 *lux*
1/ Number of significant figures of the uncertainty *2 figures rounded off to the last bigger figure.*	$u(E) = 0{,}11\ lux$ Or $U(E) = 0{,}21\ lux$
2/ Number of significant figures of the result *The last two significant figures of the result correspond to the two significant figures of the uncertainty.* *The result is rounded off to the nearest value.*	$E = 100{,}23\,464\ lux$ $E = 100{,}23\ lux$
3/ Final result *Expanded uncertainty,* $Y = (\ldots \pm U(Y))$ *level of confidence of 95%* *Do not use power of 10 notations.*	$E = 100{,}23 \pm 0{,}21\ lux\ (k=2)$ ~~$E = 100{,}23 \pm 21 \cdot 10^{-2}\ lux\ (k=2)$~~
4/ Relative uncertainty *Must be under the form of a percentage.* *(only valid if the quantity is not zero)*	$\frac{u(E)}{E} = 0{,}11\ \%$ Or $\frac{U(E)}{E} = 0{,}21\ \%$

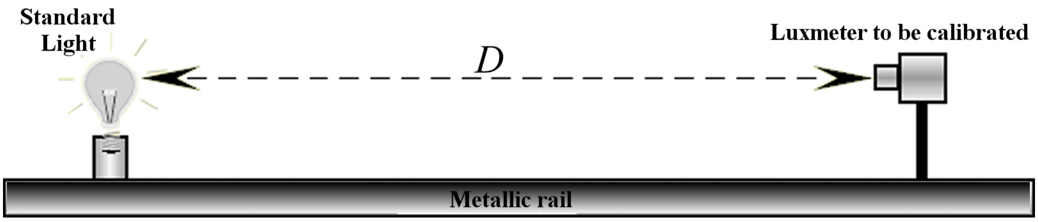
6. Procedure of evaluation of measurement uncertainty: summary



III.Calibration of a luxmeter (following [5])

The calibration method consists in comparing the value of a reference illumination with the value displayed on the luxmeter which is to be calibrated.

Figure 2 illustrates the device used for calibration.



The required device consists of:

- The required device consists of:
- a standard lamp of known luminous intensity I_L
- the luxmeter that will be calibrated.

The device provides an illumination which can be adjusted by the distance D separating the lamp from the luxmeter and which is spatially uniform on the receptive surface of the luxmeter.

1. Example of evaluation of measurement uncertainty

The operator assembles the standard lamp on one end of the bench. The luxmeter is fixed on a moveable base that can be run along the metallic rail. The operator notes down the distance D that separates the lamp from the luxmeter along with the illumination value E_{LU} *displayed by the luxmeter. The variation of the illumination level is obtained from the variation of distance* D. Repeat the operation five times .

First, the uncertainty is evaluated for the reference illumination E. This illumination is determined applying Bouguer's law. For a source considered as punctual, of intensity I_L située à la and at a distance D from the luxmeter, we can say that:

$$E = \frac{I_L}{D^2}$$

2. Uncertainty on the reference

Using the propagation of uncertainties law, express the relative standard uncertainty $u(E)/E$ in function of $u(I_L)/I_L$ and $u(D)/D$.

The model for the measurement is $E = \frac{I_L}{D^2} = f(I_L, D)$. It is a product-quotient model, hence:

$$\frac{u(E)}{E} = \sqrt{\left[\frac{u(I_L)}{I_L}\right]^2 + 4\left[\frac{u(D)}{D}\right]^2}$$

In order to determine the relative standard uncertainty $u(I_L)/I_L$ or the luminous intensity emitted by the standard lamp, we consider the information provided by its calibration certificate.

The standard lamp is calibrated with an expanded relative uncertainty of 1% $(k = 2)$. Deduce the value of the associated relative standard uncertainty $u_1(I_L)/I_L$.

With a coverage factor $(k = 2)$, we have:

$$\frac{u_1(I_L)}{I_L} = 0{,}5\ \%$$

The standard lamp shows a total range drift of 0.3% between two successive calibrations. Applying a uniform probability law, calculate the relative standard uncertainty $u_2(I_L)/I_L$ associated with this drift.

In the absence of any additional information, we consider that the time drift is associated with a uniform probability for the time intervals between 2 successive calibrations. We thus have:

$$\frac{u_2(I_L)}{I_L} = \frac{0{,}3\ \%}{2\sqrt{3}} = 0{,}087\ \%$$

The variation of the luminous intensity I_L with the supply current i does not follow a linear law. We have:

$$\frac{\Delta I_L}{I_L} = 6{,}5 \times \frac{\Delta i}{i}$$

The ammeter used to measure the current going through the lamp is calibrated with a 0.05% expanded relative uncertainty $(k = 2)$. Deduce the value of the associated relative standard uncertainty $u_3(I_L)/I_L$.

For $\Delta i = u(i)$ standard uncertainty on the current "i", we have $\Delta I_L = u_3(I_L)$ hence :

$$\frac{u_3(I_L)}{I_L} = 6{,}5\ \frac{u(i)}{i}$$

standard uncertainty on the current $\frac{u(i)}{i} = \frac{0,05 \times 10^{-2}}{2} = 2,5 \times 10^{-4}$ hence :

$$\frac{u_3(I_L)}{I_L} = 0{,}16\ \%$$

From the previously obtained results, deduce the value of the relative combined standard uncertainty $u(I_L)/I_L$.

$$\frac{u(I_L)}{I_L}=\sqrt{\left[\frac{u_1(I_L)}{I_L}\right]^2+\left[\frac{u_2(I_L)}{I_L}\right]^2+\left[\frac{u_3(I_L)}{I_L}\right]^2}$$

that is to say

$$\frac{u(I_L)}{I_L}=0{,}53\ \%=5{,}3\times 10^{-3}$$

The measurement of distance D is $3\ \mathrm{m}$ with an estimated variation interval of $\pm 5\ \mathrm{mm}$. Deduce the value of the relative uncertainty $u(D)/D$.

$$u(D)=\frac{5}{\sqrt{3}}=2{,}88\ mm$$

$$\frac{u(D)}{D}=2\frac{,88\times 10^{-3}}{3}=0{,}1\ \%=10^{-3}$$

From the previous calculations, give the value of the standard uncertainty $u_{ref}(E)$ for a reference illumination of $100\ \mathrm{lux}$.

$$\frac{u_{réf}(E)}{E}=\sqrt{\left[\frac{u(I_L)}{I_L}\right]^2+4\left[\frac{u(D)}{D}\right]^2}$$

with $\frac{u(I_L)}{I_L}=5,3\times 10^{-3}$ and $\frac{u(D)}{D}=10^{-3}$ hence

$$\frac{u_{réf}(E)}{E}=5{,}33\times 10^{-3}$$

That is to say for $E=100\ \mathrm{lux}$, $u_{ref}(E)=0.53\ \mathrm{lux}$.

3. Uncertainties associated with measurement conditions

After having dealt with the uncertainty of the reference, we will now estimate the uncertainty associated with the measurements and the luxmeter.
The illumination E_{LU} measured with the luxmeter that is to be calibrated can be modelled by:

$$E_{LU}=\bar{x}+C_{reference}+C_{resolution}$$

where

$\bar{x}$	= average of 5 replicate measurement of illumination
$C_{reference}$	= correction associated with the reference
$C_{resolution}$	= correction associated with the luxmeter resolution

The series of five measurements in similar conditions gave the following results:

N	1	2	3	4	5
E_{LU}	101	102	99	98	101

Deduce the values of $\bar{x}$ and of the estimated standard deviation $s(\bar{x}) = u_{rep}$

$$\bar{x} = 100,2\ lux$$

$$s(x) = \sqrt{\frac{1}{5-1}\sum_{k=1}^{5}(x_i - \bar{x})^2} = 1,643\ lux$$

$$s(\bar{x}) = \frac{s(x)}{\sqrt{5}} = u_{rép} = 0,735\ lux$$

Knowing that the luxmeter's resolution is of 0.5 lux for a read value of 100 lux, deduce the corresponding standard uncertainty u_{res}.

$$u_{rés} = \frac{0,5}{2\sqrt{3}} = 0,144\ lux$$

From all the previous questions, deduce the value of the combined uncertainty $u_c(E_{LU})$.

The model for the measurement is $E_{LU} = \bar{x} + C_{reference} + C_{resolution}$. Even if the corrections are null, the associated uncertainties are not. We thus have:

$$u_c^2(E_{LU}) = u_{rép}^2 + u_{réf}^2(E) + u_{rés}^2$$

$$u_c(E_{LU}) = \sqrt{u_{rép}^2 + u_{réf}^2(E) + u_{rés}^2}$$

$$u_c(E_{LU}) = \sqrt{(0,735)^2 + (0,53)^2 + (0,144)^2}$$

that is to say

$$u_c(E_{LU}) = 0,92\ lux$$

Express the measurement result of the illumination under standardized form, justifying in particular the choice of the value for the coverage constant.

We consider that the standard uncertainties u_{ref} and u_{res} are exactly known, thus they are not associated with any degrees of freedom ν_i In such conditions, the number of degrees of freedom associated with $u_c(E_{LU})$ is $\nu = 5 - 1 = 4$ which corresponds to a factor $t = 2.78$ (cf. tableau 5) (cf. table 5) of level of confidence $1 - \alpha = 0,95$ hence :

$$E_{LU} = 100,20 \pm 2,55 \; lux$$

IV.Secondary bibliography

Reference n°	Type	Title or identification
[1]	Norm	Guide pour l'expression de l'incertitude de mesure ; AFNOR X 07 020 (août 1999)
[2]	Norm	Vocabulaire international des termes fondamentaux et généraux de métrologie ; AFNOR X 07 001 (décembre 1994)
[3]	Article from a journal	WELCH B.L., J. R. Stat. Soc. Suppl. **3**, 29-48 (1938) et Biometrika **29**, 350-362 (1947)
[4]	Article from a journal	SATTERTHWHAITE F.E., Psychometrika 6, 309-316 (1946)
[5]	Book	27 exemples d'évaluation d'incertitudes d'étalonnage ; éd. Mouvement Français pour la Qualité, ISBN 2-909430-85-5 (1999)
	Book	Mesure physique et instrumentation (analyse statistique et spectrale des mesures, capteurs) ; D. BARCHIESI ; éd. ellipses TechnoSup, ISBN 2-7298-1426-4 (2003)
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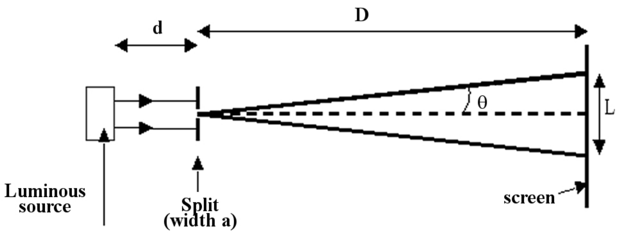
V.Exercice

1. "Measurement uncertainties": Exercise

We want to determine the width "a" of a diffracting slit from the diffraction pattern observed on a screen located at a distance "D" of the split, which is lit by a laser source of wavelength λ (cf. figure 3).



Question 1

[Solution n°1 p 27]

What is the expression of the illumination $I(\theta)$ on the screen for a split that is considered of nearly infinite length in comparison to its width "a" ?

Question 2

[Solution n°2 p 27]

Express the angle θ according to the distance between the split and the screen D and to the width L of the central lobe of diffraction measured on the screen.

Find another expression of θ from the apparent diameter of the central spot of diffraction deduced from $I$$I(\theta)$.

Deduce from these two expressions of θ the relation between the split's width a and quantities L, D and λ.

Question 3

[Solution n°3 p 27]

Using the propagation of uncertainties law, express the standard uncertainty $u(a)$.

Question 4

[Solution n°4 p 27]

The distance between the split and the screen D is measured five times in a row with a measuring tape of resolution $1\ \mathrm{mm}$. The measured values are the following:

$$\bar{D}=2{,}014\,m$$

Calculate $u(D)$.

Question 5

[Solution n°5 p 28]

The width L of the diffraction spot is measured 5 times in a row with a ruler of resolution $0.5\ \mathrm{mm}$. The measured values are the following:

measurement n°	1	2	3	4	5
measured value (mm)	25	26,5	27	23,5	24

Calculate $u(L)$.

Question 6

[*Solution n°6 p 28*]

Calculate a and $u(a)$ assuming the hypothesis that the wavelength λ of the luminous source $(\lambda = 633 \text{ nm})$ includes no uncertainty .

Question 7

[*Solution n°7 p 29*]

Calculate the expanded uncertainty $U(a)$ and express the measurement result of width a.

Solution des exercices

>Solution n°1 (exercice p. 25)

In the Fraunhofer approximation, the expression for the illumination due to the radiation diffracted by the split of width "a" is :

$$I(\theta)=I_0\frac{\left(\sin\frac{\pi\,\theta\,a}{\lambda}\right)^2}{\left(\frac{\pi\,\theta\,a}{\lambda}\right)^2}$$

>Solution n°2 (exercice p. 25)

In practice the angles θ are small values, we thus have $\tan(\theta) \approx \theta$.

As $\tan(\theta) = \frac{L}{2D} \approx \theta$, and as the apparent half diameter is $\theta = \frac{\lambda}{a}$, then we can deduce that $\frac{L}{2D} = \frac{\lambda}{a}$, hence :

$$a=\frac{2\,D\,\lambda}{L}$$

From measuring D and L, knowing λ, it is possible to determine the split's width a.

>Solution n°3 (exercice p. 25)

The model $a = f(\lambda, D, L)$ is a product-quotient model, we thus have

$$\frac{u^2(a)}{a^2}=\frac{u^2(\lambda)}{\lambda^2}+\frac{u^2(D)}{D^2}+\frac{u^2(L)}{L^2}$$

or

$$u^2(a)=\left(2\frac{D}{L}\right)^2 u^2(\lambda)+\left(\frac{2\,\lambda}{L}\right)^2 u^2(D)+\left(\frac{2\,D\,\lambda}{L^2}\right)^2 u^2(L)$$

>Solution n°4 (exercice p. 25)

We have $u(D) = \sqrt{u_{rep}^2 + u_{res}^2}$ with $u_{rep} = \frac{\sigma(D)}{\sqrt{5}}$ and $u_{res} = \frac{\delta_D}{6}$

where

$\sigma(D)$ is the standard deviation of the measured values of D.

δ_D is the resolution of the measuring instrument (Gaussian probability law in the interva l $[-0.5\text{ mm} ; +0.5\text{ mm}]$)

which gives

$$u_{rép}=\frac{0,0114}{\sqrt{5}}=0,51\,cm \quad \text{and} \quad u_{rés}=\frac{0,1}{6}=0,017\,cm$$

this is to say

$$u(D)=0,51\,cm$$

>Solution n°5 *(exercice p. 25)*

We have $u(L) = \sqrt{u_{rep}^2 + u_{res}^2}$ with $u_{rep} = \frac{\sigma(L)}{\sqrt{5}}$ and $u_{res} = \frac{\delta_L}{6}$

where

$\sigma(L)$ is the standard deviation of the measured values of L.

δ_L is the resolution of the measuring instrument (Gaussian probability law in the interval $[-0.25\text{ mm} ; +0.25\text{ mm}]$)

which gives

$$u_{rép}=\frac{1,525}{\sqrt{5}}=0,68\,mm \quad \text{and} \quad u_{rés}=\frac{0,5}{6}=0,083\,mm$$

that is to say

$$u(L)=0,685\,mm$$

>Solution n°6 *(exercice p. 26)*

The most probable values of D and L are their mean values calculated from the series of five replicate measurements, that is to say

$$\bar{D}=2,014\,m$$

$$\bar{L}=25,2\,mm$$

hence $a = \frac{2\bar{D}\lambda}{\bar{L}} = 101,18\ \mu\text{m}$

Moreover, $\frac{u(a)}{a} = \sqrt{\frac{u^2(D)}{D^2} + \frac{u^2(L)}{L^2}}$ taht is to say $\frac{u(a)}{a} = \sqrt{0,00253^2 + 0,0271^2} = 0,0273$.

Finally

$$u(a)=2,8\,\mu m$$

>Solution n°7 *(exercice p. 26)*

The coverage factor t_ν such as $U(a) = t_\nu \times u(a)$ is determined from Student's t-distribution for $\nu = 4$ degrees of freedom (five replicate measurements) that is to say $t_\nu = 2.78$ for a confidence level of 95%.

We thus have $U(a) = 7.8\ \mu\text{m}$.

Finally the result of the measurement of width a is:

$$a = 101 \pm 8\ \mu m \left(k = 2{,}78\right)$$

Glossaire

Maximum permissible error (of a measuring instrument)

Extreme values of measurement error, permitted by specifications, regulations, etc., for a given measuring instrument.

Measurement uncertainty

Parameter associated with the result of a measurement, characterizing the dispersion of the quantity values which can reasonably be attributed to the measurand.

Random error of measurement

Result of a measurement minus the average from an infinite number of measurements of the same measurand, when these measurements are performed in repeatability conditions.

Repeatability

Closeness of agreement between the successive measurement results for the same measurand, with these measurements performed in the exact same measurement conditions.

These conditions are called r"epeatability conditions", and include:

- the same operating procedure,
- the same observer,
- the same measuring instrument used in the same conditions,
- replication of measurements within a short period of time.

Systematic measurement error

Average which would result of an infinite number of measurements of the same measurand when these measurements are performed in repeatability conditions, minus a true value of the measurand.

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