

# Optical metrology through fractional Fourier transforms

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# I.Présentation

*Module :*

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Optical metrology

*Auteur(s) :*

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*Résumé :*

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*Mots-clés :*

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*Pré-requis :*

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*Objectif(s) pédagogique(s) :*

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*Plan du cours :*

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- The detection of linear frequency chirped functions
  - Propagation in a clear space and fractional Fourier transforms
  - Fiber holograms analyzed by fractional *FTs*
  - Conclusion

*Conception & production :*

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# II.Course

## 1. The detection of linear frequency chirped functions

### 1.1. Linear frequency

We want here to point out the usefulness of fractional Fourier transforms to analyse linear frequency chirped signals also called chirps. These signals are characterized by the linear evolution of their spectral contents. The harmonic signal  $u(x)$  is defined by:

$$u(x) = \exp(i 2 \pi \chi x). \quad (1.1)$$

And its instant frequency is:

$$f_i(x) = \frac{1}{2\pi} \frac{\partial \varphi(x)}{\partial x} = \chi, \quad (1.2)$$

Where  $\varphi(x)$  is the phase of  $u(x)$ . The signal  $u(x)$  is said to be stationary if for any  $x$  the instantaneous frequency  $f_i(x)$  is constant [Fla98 [[Fla98]]]. The standard Fourier transform is well adapted to analyse such signals. The *FT* of  $u(x)$  allows us to point out its spectral properties synthetically. This function is a Dirac impulsion centred on the  $\chi$  frequency.

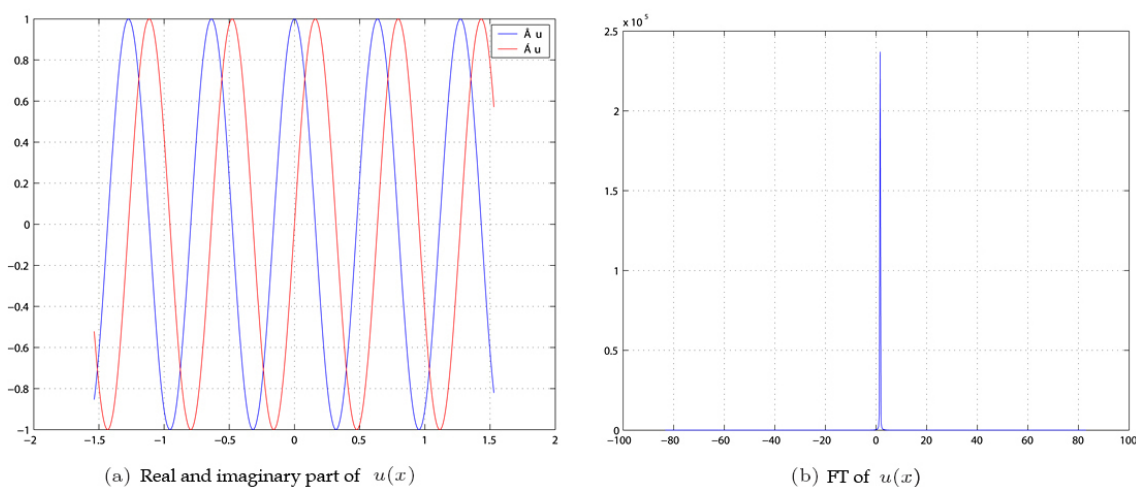


Figure 1

\* Figure 1: Representation of  $u(x) = \exp(i 2 \pi \chi x^2)$ . and its FT.

Figures 1(a) and (b) represent  $u(x)$  and its FT. If the function  $u(x)$  is the linear frequency chirped function so that:

$$u(x) = \exp(i 2 \pi \chi x^2). \quad (1.3)$$

Its fractional Fourier transform is equal to:

$$F_{\alpha}[u(x)] = \sqrt{\frac{1+i \tan \chi}{1+2 \chi \tan \alpha}} \exp\left[i \pi x^2 \frac{2 \chi - \tan \alpha}{1+2 \chi \tan \alpha}\right]. \quad (1.4)$$

For  $\alpha = \frac{\pi}{2}$  we find the standard Fourier transform which is again a chirped function.

Conversely, for  $\alpha = \alpha_{opt} = \arctan\left(-\frac{1}{2\chi}\right)$  and when using equality [H.01 [[G.78]]]:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\pi i \varepsilon}} \exp\left(-\frac{x^2}{i \varepsilon}\right) = \delta(x), \quad (1.5)$$

We obtain:

$$F_{\alpha_{opt}}[u(x)] = \delta(x). \quad (1.6)$$

This time the Dirac distribution is centred on the origin of the coordinates. The function  $u(x)$  is therefore a centred function.

### Remarque

The fractional order defines the frequency drift. Fractional Fourier transforms allow us to detect linear chirps in a signal.

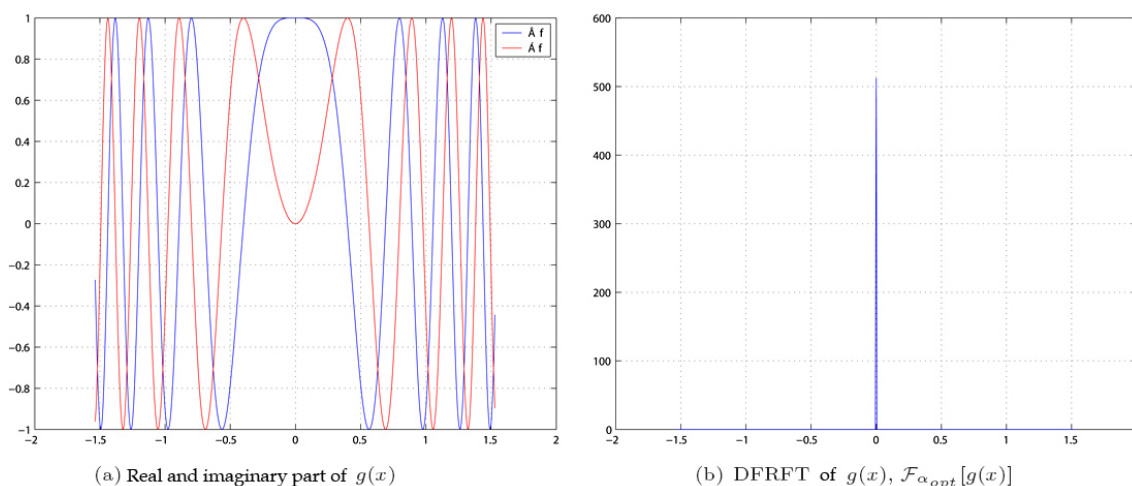


Figure 2

\* Figure 2: Representation of  $g(x) = \exp(i2\pi\chi x^2)$  and its optimal order FFT.

Figures 2(a) and 2(b) illustrate this property. Besides, this property will allow us to conduct a metrology since diffraction phenomena in Fresnel's approximation have a linear frequency drift.

## 2. Propagation in a clear space and fractional Fourier transforms

### 2.1. Propagation equations solved by standard FTs

First of all, we shall consider the case where the wave of amplitude  $U$  propagates between two planes separated by a distance  $z$  as illustrated in figure 3.

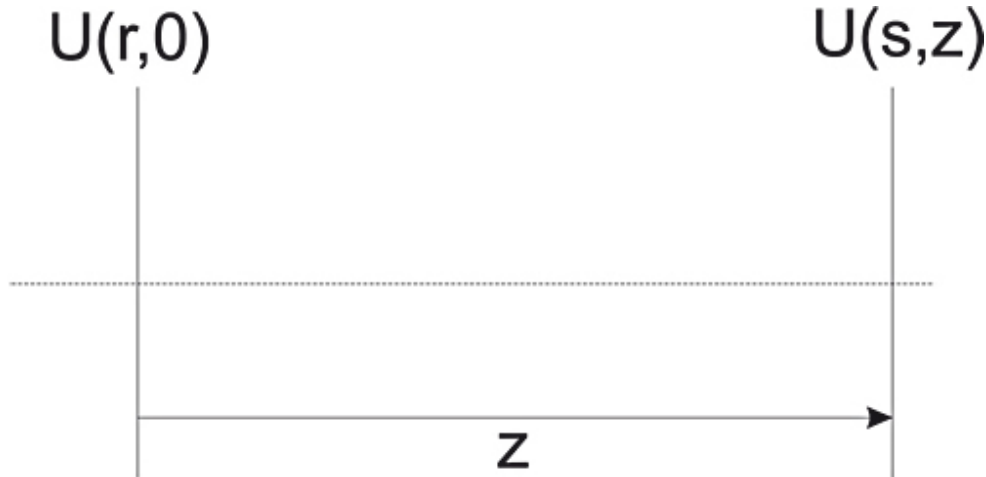


Figure 3 : Wave propagation between two planes separated by a distance  $z$

The left plane is the reference plane situated on  $z = 0$ . It comprises the amplitude of the field noted  $U(r,0)$  with  $r = (\xi, \eta)$ . The right plane is the target plane situated at a distance  $z$  from the reference plane. It contains the amplitude of the field noted  $U(s,z)$  with  $s = (x, y)$ .

#### Rappel

The propagation wave between two planes in a clear space is controlled by the following differential equation [Sie86 [[H. 01]]]:

$$\nabla_{\perp}^2 U(s, z) + i 2k \frac{\partial U(s, z)}{\partial z} = 0, \quad (2.1)$$

Where  $\perp$  defines the transversal field of  $U(s, z)$ ,  $k = 2\pi/\lambda$  is the wave number and  $\lambda$  the wave length.

To solve the differential equation, we use the standard Fourier transformation. The transformation is defined by:

$$\mathbf{F}[f(s)](\sigma) = \hat{f}(\sigma) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(s) \exp[-i 2\pi \sigma \cdot s] ds, \quad (2.2)$$

With  $\sigma = (u, v)$  and its norms is  $\sigma = \sqrt{u^2 + v^2}$ .  $u$  and  $v$  are the spatial frequencies following the two coordinate axes,  $d\sigma = dudv$ . The scalar product  $\sigma, s$  gives  $(ux + vy)$ . Differential equation (2.1) in the spectral domain becomes:

$$i2k \frac{\partial \hat{U}(\sigma, z)}{\partial z} = 4\pi^2 \sigma^2 \hat{U}(\sigma, z). \quad (2.3)$$

In the spectral domain, the differential equation of second order for spatial variables  $(x, y)$  and first order for variable  $z$  is transformed into a first order equation 1 for propagation variable  $z$ . Its equation can be solved such that:

$$\hat{U}(\sigma, z) = \hat{U}(\sigma, 0) \exp[-i\pi\lambda z \sigma^2]. \quad (2.4)$$

The expression of  $U(s, z)$  in the spatial domain is determined by an inverse transformation of  $\hat{U}(\sigma, z)$ , that is to say:

$$U(s, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{U}(\sigma, z) \exp[i2\pi s \cdot \sigma] d\sigma. \quad (2.5)$$

By putting equation (2.4) in (2.5), we obtain:

$$U(s, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{U}(\sigma, 0) \times \exp[-i\pi\lambda z \sigma^2] \exp[i2\pi s \cdot \sigma] d\sigma. \quad (2.6)$$

Thus, we obtain here a Fourier transformation from the product of two functions in  $\sigma$ . In the spatial domain, it gives a convolution product. As:

$$\int_{-\infty}^{+\infty} \exp[-p^2 u^2 \pm qu] du = \frac{\sqrt{\pi}}{p} \exp\left[\frac{q^2}{4p^2}\right], \quad (2.7)$$

we obtain the value of the field after propagation so that:

$$U(s, z) = U(s, 0) * \left[ \frac{1}{i\lambda z} \exp\left(\frac{i\pi}{\lambda z} s^2\right) \right]. \quad (2.8)$$

The symbol  $*$  represents the bidimensional convolution for spatial variables. The integral form of relation (2.8) is well known:

$$U(s, z) = \frac{1}{i\lambda z} \iint U(r, 0) \cdot \exp\left[\frac{i\pi}{\lambda z} (r-s)^2\right] dr. \quad (2.9)$$

We often find the expression in its developed form so that:

$$U(s, z) = \frac{1}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z} s^2\right] \iint U(r, 0) \exp\left[\frac{i\pi}{\lambda z} r^2\right] \exp\left[-i\frac{2\pi}{\lambda z} r \cdot s\right] dr. \quad (2.10)$$

### Remarque

In this paragraph the standard FT was used to solve the propagation equation.

### Attention

**The mathematical structure of Fresnel integrals has nearly the same structure as Fractional FTs. So, there is no reason why propagation integrals cannot be expressed using Fractional FTs.**

The following paragraph will explain this method.

## 2.2. Propagation described by fractional FTs

### a) Dimensioning variables

#### Définition

The mathematical definition of fractional Fourier transforms is:

$$F_\alpha[f(r)](s_a) = C(\alpha) \exp\left(i\frac{\pi s_a^2}{\tan \alpha}\right) \int_{-\infty}^{+\infty} f(r) \exp\left(i\frac{\pi r^2}{\tan \alpha}\right) \exp\left(-i2\pi\frac{rs_a}{\sin \alpha}\right) dr, \quad (2.11)$$

with:

$$C(\alpha) = \frac{\exp[i\alpha]}{i \sin \alpha}, \quad (2.12)$$

where  $\alpha > 0$ .

As it can be noted, if we want to describe propagation through spatial variables, the definition of fractional *FTs* is problematic because its variables have no dimension. This is the first problem. A solution was offered by [Loh93 [[Loh93]]]. This solution consists in optically interpreting fractional *FTs* in Wigner space.

The optical elements of this interpretation need to be found.

#### Rappel

The  $\alpha$  order fractional *FT* of a function induces a clockwise  $\alpha$  angle rotation to its Wigner distribution.

Lohmann divided this rotation in three steps as illustrated in figure 4.

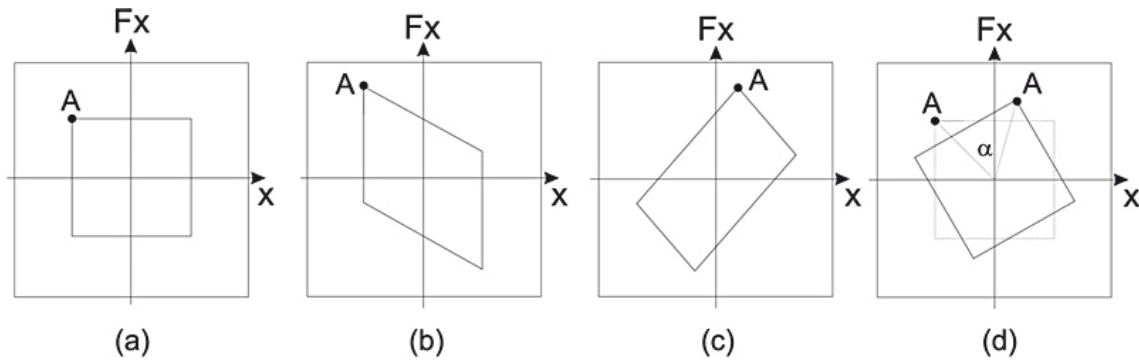


Figure 4

Figure 4: (a) reference outline, (b) frequency stretch, (c) special stretch, (d) frequency stretch.  $x$ , spatial variable ;  $F_x$ , frequency variable.

This figure is for a  $\pi/3$  angle  $\alpha$ .

These 3 steps consist of the frequency stretch of the distribution, then a spatial stretch and finally another frequency stretch. Frequency stretches are optically obtained through a lens and the spatial stretch through a convolution product, in other words propagation. In [Loh93 [[Loh93]]], parameter  $\epsilon$  appears in the form of an abstract focal  $f_1$  without physical direction. This ambiguity was found by replacing parameter  $\lambda f_1$  by  $\epsilon^2$  when interpreting fractional  $FTs$  by a Wigner distribution.

The interpretation of fractional  $FTs$  can be expressed as:

$$F_{\alpha}[f(r)](s) = C(\alpha) \exp\left(i \frac{\pi s^2}{\epsilon^2 \tan \alpha}\right) \int_{-\infty}^{+\infty} f(r) \exp\left(i \frac{\pi r^2}{\epsilon^2 \tan \alpha}\right) \exp\left(-i 2 \pi \frac{rs}{\epsilon^2 \sin \alpha}\right) dr \quad (2.13)$$

where variables  $[s],[r]$  and  $[\epsilon]$  have the dimension of a length. The normalization coefficient  $C(\alpha)$  equals to:

$$C(\alpha) = \frac{\exp[i\alpha]}{i \epsilon^2 \sin \alpha}. \quad (2.14)$$

For  $\epsilon = 1$  and variables without dimension, the mathematical definition of fractional Fourier transforms is found again. In our case, the definition given to  $\epsilon^2$  is linked to the way we sample the signal and to the expression of the digital  $FT$ . In fact, digital technology is employed in metrology using fractional  $FTs$ . To define the value of  $\epsilon$ , we have to write the definition of fractional  $FTs$  where  $\alpha = \pi/2$ :

$$F_{\pi/2}[f(r)](s) = C(\pi/2) \int_{-\infty}^{+\infty} f(r) \exp\left[-i 2 \pi \frac{rs}{\epsilon^2}\right] dr. \quad (2.15)$$

Its discrete version equals to:

$$F_{\pi/2}[f(m, n)](k, l) = C(\pi/2) \sum_{m=-M/2}^{M/2-1} \sum_{n=-N/2}^{N/2-1} f(m, n) \exp\left[-i 2 \pi \left(\frac{mk \delta_x \delta_x}{\epsilon^2} + \frac{nl \delta_y \delta_y}{\epsilon^2}\right)\right] \delta_x \delta_y \quad (2.16)$$

Where  $\delta_\xi$  and  $\delta_x$  refer to the sampling interval of  $f(r)$  and its transform on the  $x$ -axis, and  $\delta_\eta$  and  $\delta_y$  on the  $y$ -axis.

The number of samples is  $M$  on the  $x$ -axis and  $N$  on the  $y$ -axis.

No matter if we are in the spatial or spectral domain, the number of samples is the same.

Relation (2.16) has to be written as a discrete computer-generated Fourier transform, namely:

$$\mathbf{F}_{\pi/2}[f(m, n)](k, l) = C(\pi/2) \sum_{m=-M/2}^{M/2-1} \sum_{n=-N/2}^{N/2-1} f(m, n) \exp\left[-i 2\pi \left(\frac{mk}{M} + \frac{nl}{N}\right)\right] \delta_\xi \delta_\eta \quad (2.17)$$

By identifying relations (2.17) and (2.16), we obtain:

$$\frac{\delta_\xi \delta_x}{\epsilon^2} = \frac{1}{M} \quad \text{and} \quad \frac{\delta_\eta \delta_y}{\epsilon^2} = \frac{1}{N} \quad (2.18)$$

Digital  $FT$ s only "see" pixels without dimension, just like  $FFT$ s (Fast Fourier Transforms).

Therefore, no matter the description domains used (spatial or frequency), pixels are the sampling unit.

That is why we assume  $\delta_\xi = \delta_x$  and  $\delta_\eta = \delta_y$ .

### Attention

**Generally speaking, pixels are identical on the two axes of the description plane and for any description domain. So, the sampling interval chosen equals to  $\delta$ .**

**Most of the time, the number of pixels on both axes is the same:  $N = M$ . If this is not the case, the dimensions of the image are cut off to obtain a square image. We eventually obtain:**

$$\epsilon^2 = N \delta^2 \quad (2.19)$$

**If a square dimension image cannot be obtained, two  $\epsilon$  are defined such that  $\epsilon = (\epsilon_\xi, \epsilon_\eta) = (M, N)\delta^2$ .**

### b) Residual bends

As Fourier transformations are a special case of fractional  $FT$ s, we can write:

$$\mathbf{F}_{\pi/2}[f(r)](s) = \frac{1}{\epsilon^2} \int \int_{-\infty}^{+\infty} f(r) \exp\left(-i 2\pi \frac{rs}{\epsilon^2}\right) dx. \quad (2.20)$$

From this definition of  $FT$ s, equation (2.4) becomes:

$$\hat{U}(s, z) = \hat{U}(s, 0) \exp\left[-i \pi \lambda z \frac{s^2}{\epsilon^4}\right]. \quad (2.21)$$

$U(\mathbf{r}, z)$  is the inverse Fourier transform of  $\hat{U}(\mathbf{s}, z)$ :

$$U(r, z) = C(-\pi/2) \int \int_{-\infty}^{+\infty} \hat{U}(s, 0) \exp\left[-i\pi\lambda z \frac{s^2}{\epsilon^4}\right] \exp\left(-i2\pi \frac{rs}{\epsilon^2 \sin(-\pi/2)}\right) ds, \quad (2.22)$$

which can be also described as an fractional Fourier transform of  $\alpha$  order such that:

$$\hat{U}(s_y, z) = \frac{C(-\pi/2)}{C(\alpha)} \exp\left(-i \frac{\pi s_y^2}{\epsilon^2 \tan \alpha}\right) F_\alpha[\hat{U}(s, 0)](s_y), \quad (2.23)$$

where:

$$\hat{U}(s_y, z) = U\left(-\frac{s_y}{\sin \alpha}, z\right), \quad s_y = -s \sin \alpha, \quad (2.24)$$

and of fractional order:

$$\cot \alpha = \frac{-\lambda_z}{\epsilon^2}. \quad (2.25)$$

As the distribution of the field  $\hat{U}(\mathbf{s}, 0)$  is the  $\pi/2$  order fractional Fourier transform of  $U(r, 0)$ , using composition  $F_\alpha \circ F_{\pi/2} = F_{\alpha+\pi/2}$ , we can define the amplitude distribution of the field at a distance  $z$  following the amplitude distribution at the distance  $z = 0$  by factorizing a  $\gamma = \alpha + \pi/2$  order fractional Fourier transform with a quadratic phase term such that:

$$\hat{U}(s_y, z) = \cos \gamma \exp[-i\gamma] \exp\left(i \frac{\pi s_y^2}{\epsilon^2 \tan \gamma}\right) F_\gamma[U(r, 0)](s_y) \quad (2.26)$$

According to Bonnet and Pellat-Finet [P.94 [[P.94]]] [Pel.94 [[Pel94]]], relation (2.26) emphasises the problem of quadratic phase residual depending on variable  $s$ . When the diffraction in Fraunhofer approximation is considered, a standard Fourier transform is enough. In this approximation, quadratic phase terms are not involved. In other words, bends are not involved either. Only planes are considered.

### Remarque

When Fresnel approximations are considered, we noted that fractional Fourier transforms can be used, but this interpretation leaves a quadratic phase term. Thus, this is not a standard fractional *FT*. The notion of plane is not adapted to analyse imaging systems in coherent light quantity.

The metaxial theory seems to be an outcome of the paraxial optics theory.

Bonnet clearly explains his approach:

*Research on coherent imaging requires that equation (2.26) does not contain the term  $s$  which affects it. This quadratic phase term refers to the notion of transparent bend. This transparent*

bend becomes neutral for a special bend [G.78 [[H. 01]]]. **Therefore, we find a coherent image on a spherical sensor.**

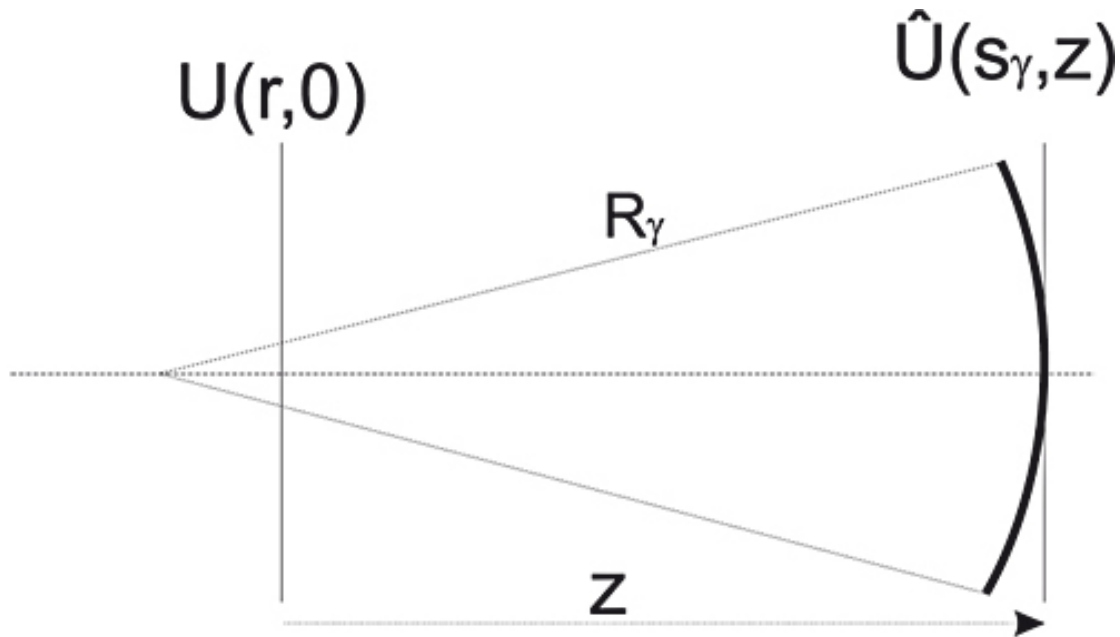


Figure 5

\* Figure 5: Coherent image on the portion of a sphere with a radius of curvature  $R_\gamma = \epsilon^2 / (\lambda \tan \gamma)$ .

Thus, the coherent image of  $U(r,0)$  can be found for a sphere with a radius of curvature  $R_\gamma = \epsilon^2 / (\lambda \tan \gamma)$  as illustrated in figure 5.

The field for the portion of a sphere is defined by:

$$\hat{U}(s_\gamma, z) = \cos \gamma \exp(-i\gamma) \mathbf{F}_\gamma[U(r,0)](s_\gamma) \quad (2.27)$$

with the following scale changes:

$$\hat{U}(s_\gamma, z) = U\left(\frac{s_\gamma}{\cos \gamma}, z\right), \quad s_\gamma = s \cos \gamma \quad (2.28)$$

The  $\gamma$  fractional order transformation involves a change in relation (2.25):

$$\tan \gamma = \frac{\lambda z}{\epsilon^2} \quad (2.29)$$

The properties of fractional  $FT$ s become interesting when describing propagation.

If  $z$  tends to 0, the  $\gamma$  order is null. So, the fractional operator tends to  $\mathbf{F}_0$  which is the identity operator.

Now, if  $z$  tends to infinity,  $\gamma$  tends to  $\pi/2$ . This is the standard  $FT$  operator.

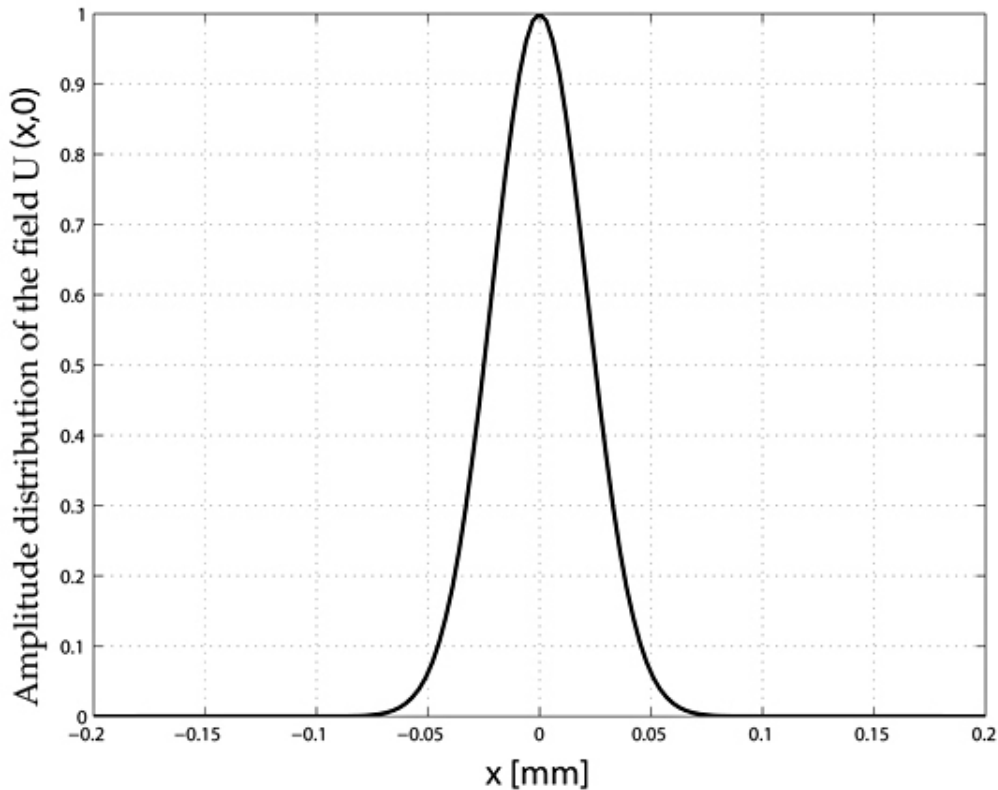


Figure 6

\* Figure 6: (a) Gaussian function at  $z = 0$  with  $\omega_0 = 30 \mu\text{m}$ ,  $\lambda = 633 \text{ nm}$ .

As far as diffraction is concerned, this is diffraction in Fraunhofer approximation.

Therefore, there is a continuum between the reference field and its angular spectrum.

### Attention

**The theoretical notion of spherical sensor does not present any particular problem in most metrology methods. A sensor is a quadratic sensor only sensitive to intensity. The phase term in  $s_y^2$  of equation (2.26) is not involved.**

### c) Example

If we take a Gaussian beam with an amplitude defined by:

$$U(r,0) = \exp\left[-\frac{r^2}{\omega_0^2}\right], \quad (2.30)$$

with  $\omega_0$  the radius of the beam in  $1/e$ . Figure (6) provides a graphical representation.

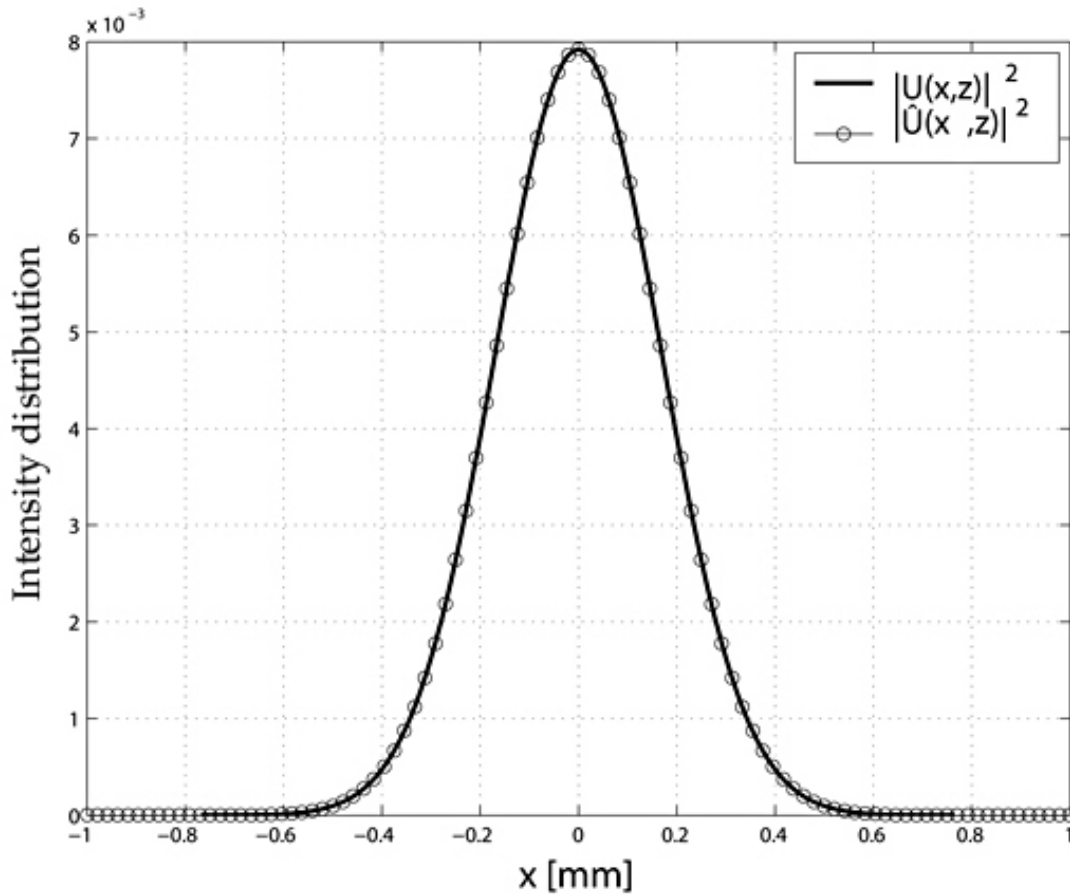


Figure 7

\* Figure (7): intensity distributions in  $z = 50 \text{ mm}$  with  $\omega_0 = 30 \mu\text{m}$ ,  $\lambda = 633 \text{ nm}$ .

The amplitude distribution of the Gaussian beam at a distance  $z$  given Fresnel relation (2.10) equals to:

$$U(r, z) = \frac{1}{1 + i \left( \frac{\lambda z}{\pi \omega_0^2} \right)} \exp \left[ -\frac{r^2}{\omega^2(z)} \right] \exp \left[ i \pi \frac{r^2}{\lambda R(z)} \right], \quad (2.31)$$

with:

$$\omega(z) = \frac{\lambda z}{\pi \omega_0} \left[ 1 + \left( \frac{\pi \omega_0^2}{\lambda z} \right)^2 \right]^{1/2}, \quad \text{and} \quad R(z) = z \left[ 1 + \frac{\left( \frac{\pi \omega_0^2}{\lambda z} \right)^2}{1 + \left( \frac{\pi \omega_0^2}{\lambda z} \right)^2} \right]^{-1}. \quad (2.32)$$

The comparison between the intensity distributions  $|U(\mathbf{r}, z)|^2$  and  $|\hat{U}(s_\gamma, z)|^2$ , with  $\omega_0 = 30 \mu\text{m}$ ,  $\lambda = 633 \text{ nm}$  and  $z = 50 \text{ mm}$ , is illustrated in figure (7).

The  $\gamma$  fractional order equals to  $1.426 \text{ rad}$ , and amplitude distribution  $\hat{U}(s_\gamma, z)$  is found for a sphere with a radius of curvature  $R_\gamma = 1.0598 \text{ mm}$ .

\* \*  
\*

With the definition of  $\epsilon^2$ , we synthesized the mathematical and optical definitions of FRFTs and the digitization of optical signals using a system of digital acquisition.

### 3. Fiber holograms analyzed by fractional FTs

The direct analysis of diffraction fringes can resort to different methods of digital restitution, for example inverse Fresnel transformation.

This method prevents us from finding the shape of the object back when the coding beam is astigmatic: curvature radii in two orthogonal axes  $x$  and  $y$  are different and nothing can invite us to choose different quadratic phases in Fresnel core.

Besides, fractional Fourier Transforms enable us to choose different quadratic phases following both axes.

The intensity distributions of a diffracted field by objects (fibers, particles, etc.) recorded on a quadratic medium in in-line-holography will involve different linear frequency drifts along the axes  $(x, y)$  of the image.

Nevertheless, we consider here that the beam is flat and has no aberration. This section offers a method to analyze fiber holograms by fractional Fourier transforms. The approach within fractional  $TF$  metrology is first to define the intensity distribution in the quadratic sensor plane and, secondly, to identify the functions having linear frequency drifts to analyze them using fractional  $FT$ s.

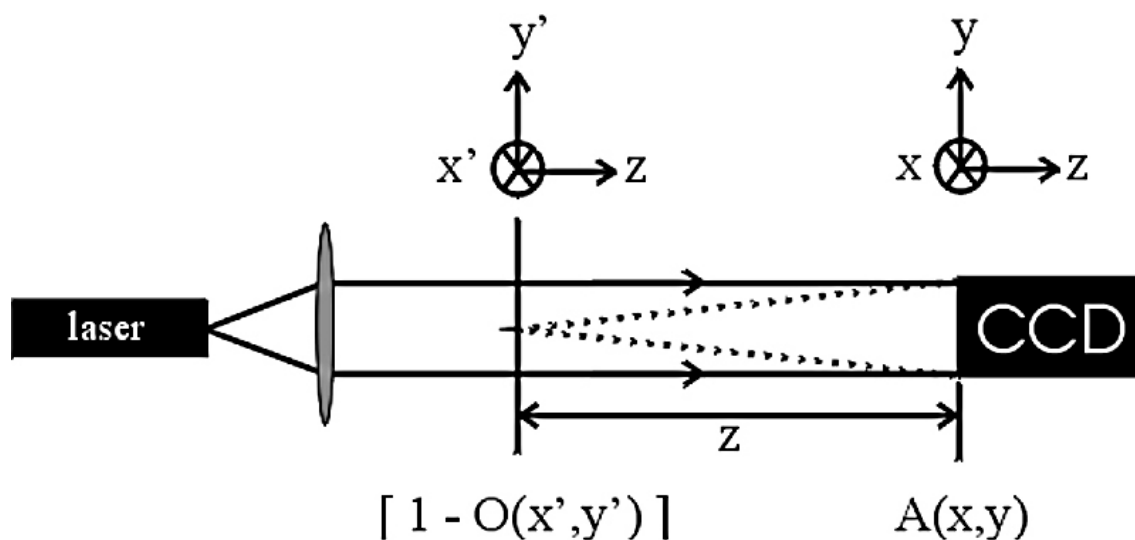


Figure 8

### 3.1. Equations of fiber holograms

#### a) Calculation of the fiber-diffracted field

We consider an opaque object, noted  $[1 - \mathcal{O}(x', y')]$  located at a distance  $z$  from the CCD sensor, as shown in figure 8. According to Huygens integral and following Fresnel approximation, the amplitude of the complex field, noted  $A(x, y)$  is given by:

$$A(x, y) = \frac{\exp(i \frac{2\pi}{\lambda} z)}{i \lambda z} \int_{-\infty}^{+\infty} [1 - \mathcal{O}(x', y')] \exp\left(\frac{i\pi}{\lambda z} [(x-x')^2 + (y-y')^2]\right) dx' dy' \quad (3.1)$$

In the particular case where the transmittance of the object is expressed by a one-dimension function, equation (3.1) is expressed in a simplified form:

$$A(x) = \frac{\exp\left(i \frac{2\pi z}{\lambda}\right)}{\sqrt{\lambda z}} \int_{-\infty}^{+\infty} [1 - \mathcal{O}(x')] \exp\left(\frac{i\pi}{\lambda z} \left[(x-x')^2 - \frac{\pi}{4}\right]\right) dx' \quad (3.2)$$

For an opaque fiber of diameter  $d$  centered in the origin, the transmittance of such an object is defined by the following equation:

$$\mathcal{O}(\xi) = \text{rect}\left[\frac{\xi}{d}\right] = \begin{cases} 1 & \text{if } |\xi| < d/2, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

The complex field  $A(x)$  is expressed using first and second level Fresnel integrals:

$$A(x) = 1 - \frac{\exp(-i \frac{\pi}{4})}{\sqrt{2}} \cdot \{ \mathcal{E}(\mu_1) - \mathcal{E}(\mu_2) + i[\mathcal{S}(\mu_1) - \mathcal{S}(\mu_2)] \}, \quad (3.4)$$

where  $\mu_1 = \sqrt{\frac{2}{\lambda z}}(x + d/2)$  and  $\mu_2 = \sqrt{\frac{2}{\lambda z}}(x - d/2)$ .  $\mathcal{C}(x)$  and  $\mathcal{P}(x)$  are first and second level Fresnel integrals. Their definition is:

$$\mathcal{E}(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad \mathcal{S}(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt. \quad (3.5)$$

In the far field approximation, i.e. when  $z > 10d^2/\lambda$ , the intensity distribution is expressed by the following equation:

$$I(x) = 1 - \frac{2}{\sqrt{\lambda z}} \cos \left[ \frac{\pi x^2}{\lambda z} - \frac{\pi}{4} \right] F(x) + \frac{1}{\lambda z} F^2(x) \quad (3.6)$$

Figure 9 gives an example of intensity distribution.

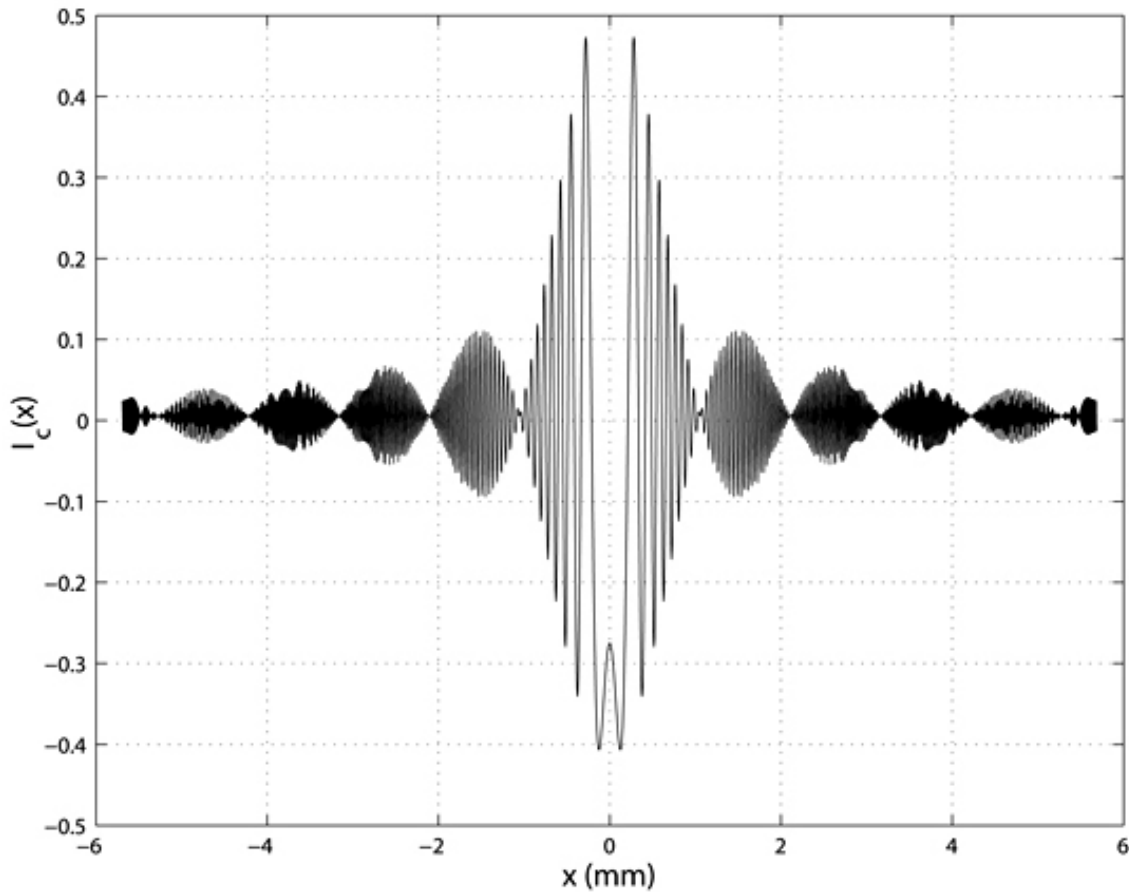


Figure 9

\* Figure 9: Representation of the intensity distribution  $I(x)$  for  $z = 100 \text{ mm}$  and  $d = 60 \mu\text{m}$ .

$$F(x) = d \sin c \left[ \frac{dx}{\lambda z} \right] = \frac{\sin \left( \frac{\pi dx}{\lambda z} \right)}{\frac{\pi dx}{\lambda z}} \quad (3.7)$$

The expression is the Fourier transform of  $O(x')$ . As shown in equation (3.6), Intensity  $I(x)$  is partially described by a real linear chirped function:

$$\cos \left[ \frac{\pi x^2}{\lambda z} - \frac{\pi}{4} \right]. \quad (3.8)$$

We will show that it is possible to analyze this type of signal by fractional *FTs*. As far field approximations are less restrictive than near-field approximations, we first focus on equation (3.6). The results remain valid in the near field.

As noted in the lesson or the previous chapter, the most important property of fractional Fourier transforms is that which connects them to Wigner distribution functions. Therefore, in the following paragraph, we will first examine the Wigner distribution function of the second term of equation (3.6).

### b) Wigner distribution functions of a rectangular opening

We should keep in mind that Wigner distribution functions are defined by the integral form:

$$W_{S,S}(x, \nu) = W_S(x, \nu) = \int_{-\infty}^{+\infty} S\left(x + \frac{\xi}{2}\right) \cdot S\left(x - \frac{\xi}{2}\right) \exp(-i 2 \pi \nu \xi) d \xi, \quad (3.9)$$

We shall examine the case of a function  $S(x)$  defined by:

$$S(x) = \cos\left[\frac{\pi x^2}{\lambda z} - \frac{\pi}{4}\right] F(x). \quad (3.10)$$

The function  $S(x)$  is the sum of two exponential terms:

$$S(x) = \frac{1}{2} \exp\left(-i \frac{\pi}{4}\right) S_+(x) + \frac{1}{2} \exp\left(i \frac{\pi}{4}\right) S_-(x), \quad (3.11)$$

with:

$$\begin{aligned} S_+ &= \exp\left(+i \frac{\pi x^2}{\lambda z}\right) F(x) \\ S_- &= \exp\left(-i \frac{\pi x^2}{\lambda z}\right) F(x) \end{aligned} \quad (3.12)$$

Wigner distributions are bilinear. Therefore, a nonlinear term will appear (called interferences) in the case of a multi-component signal. The Wigner distribution of (3.10) is finally:

$$W_S(x, \nu) = \frac{1}{4} \left[ W_{S_+}(x, \nu) + W_{S_-}(x, \nu) \right] + N(x, \nu). \quad (3.13)$$

Using the method of stationary phases, the development of a nonlinear term, noted  $N$ , gives the following result:

$$N(x, \nu) = k \sin c \left[ \frac{\pi d}{\lambda z} (x + 2 \lambda z \nu) \right] \sin c \left[ \frac{\pi d}{\lambda z} (x - 2 \lambda z \nu) \right] \sin \left[ \frac{2 \pi}{\lambda z} x^2 - 2 \pi \lambda z \nu^2 + \frac{\pi}{4} \right] \quad (3.14)$$

with  $k = -\sqrt{\frac{\lambda z}{4\pi}}$ . We clearly see that interference terms contain all information related to the system, that is to say the information about the axial position  $z$  and the diameter  $d$  of the fiber.

However, a study on this part of the signal does not seem trivial to describe diffraction patterns.

### Remarque

If an energetic profile is desired, its contribution is weakened by the fact that it takes positive and negative values.

Therefore, we focus on the first two terms of equation (3.13).

Considering that  $F(x)$  is the Fourier transform of  $O(x')$  and that relations (2.32), (2.33) and (2.34) of the first lesson on the bases of fractional *FTs* and their properties to Wigner distributions, the first two terms of equation (3.13) can be written as the sum of the following Wigner distributions:

$$\mathbf{W}_{S_+}(x, \nu) + \mathbf{W}_{S_-}(x, \nu) = \mathbf{W}_O\left(-\nu + \frac{x}{\lambda z}, x\right) + \mathbf{W}_O\left(-\nu - \frac{x}{\lambda z}, x\right) \quad (3.15)$$

where:

$$\mathbf{W}_O(x, \nu) = (d - |x|) \operatorname{sinc}[\nu(d - |x|)] \operatorname{rect}\left[\frac{x}{d}\right] \quad (3.16)$$

$\mathcal{W}_O(x, \nu)$  is the distribution function of the rectangular function defined by (3.3). Figure 10 illustrates the Wigner distribution of  $\mathcal{S}(x)$ .

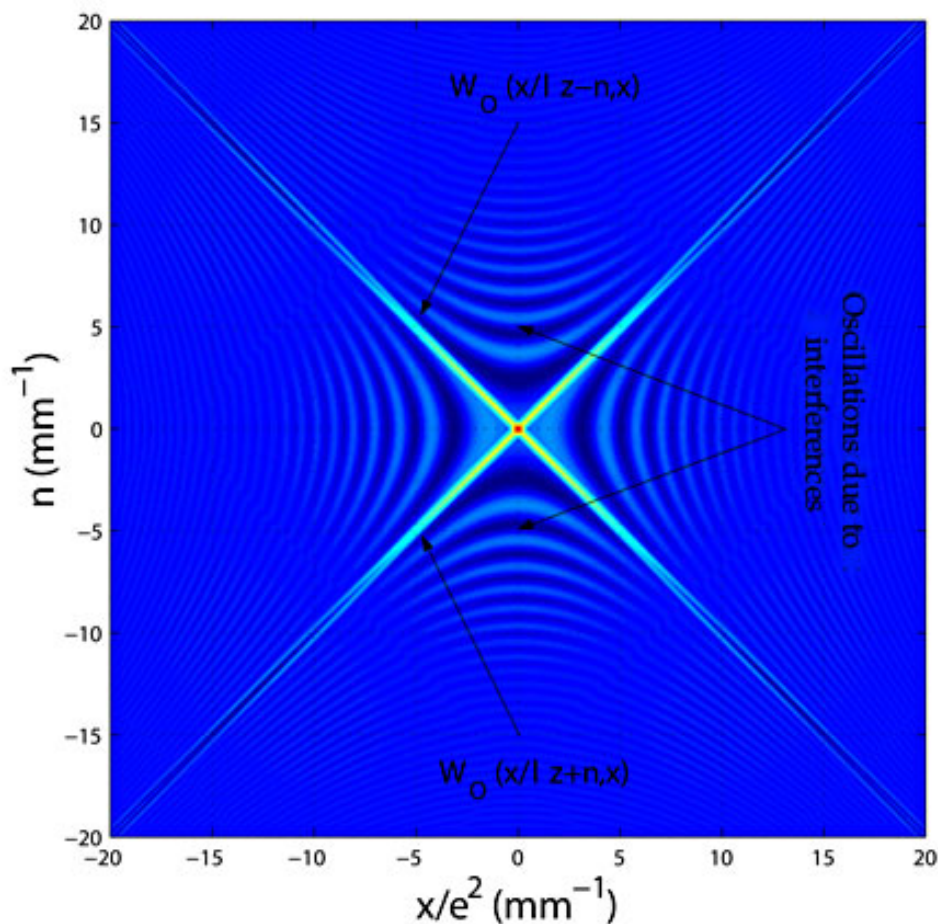


Figure 10

\* Figure 10: Representation of the Wigner distribution of  $\mathcal{S}(x)$  for  $z = 100 \text{ mm}$  and  $d = 60 \mu\text{m}$   $\epsilon^2 = N\delta^2$ ,  $N$  is the number of samples and  $\delta$  the sampling interval.

In this example, the distance  $z$  chosen equals to  $100 \text{ mm}$  and the diameter  $d$  equals to  $60 \mu\text{m}$ .

This graph indicates the two distributions  $\mathcal{W}_s(x, \nu)$  and  $\mathcal{W}_s(x, \nu)$  and also the four external interference zones linked to the bilinearity of the operator.

The two branches of distribution are centered on variation laws  $\nu = x/\lambda z$  and  $\nu = -x/\lambda z$ .

**Wigner distribution keeps the opening of the function on which it is applied.**

Therefore the width of every branch corresponds to the diameter  $d$  of the opening of the rectangular function ( $\text{rect}(x/d)$ ).

The intensity distribution  $I(x)$  (equation (3.6)) was traced in figure 9. In this case, the constant was removed.

This operation allows us to clearly indicate the Wigner distribution of the useful signal.

In fact, in Wigner space, a constant is translated by a Dirac pulse which prevents any signal description.

The representation in figure 9 is associated with its representation in Wigner space noted  $\mathcal{W}_I$  (figure 11).

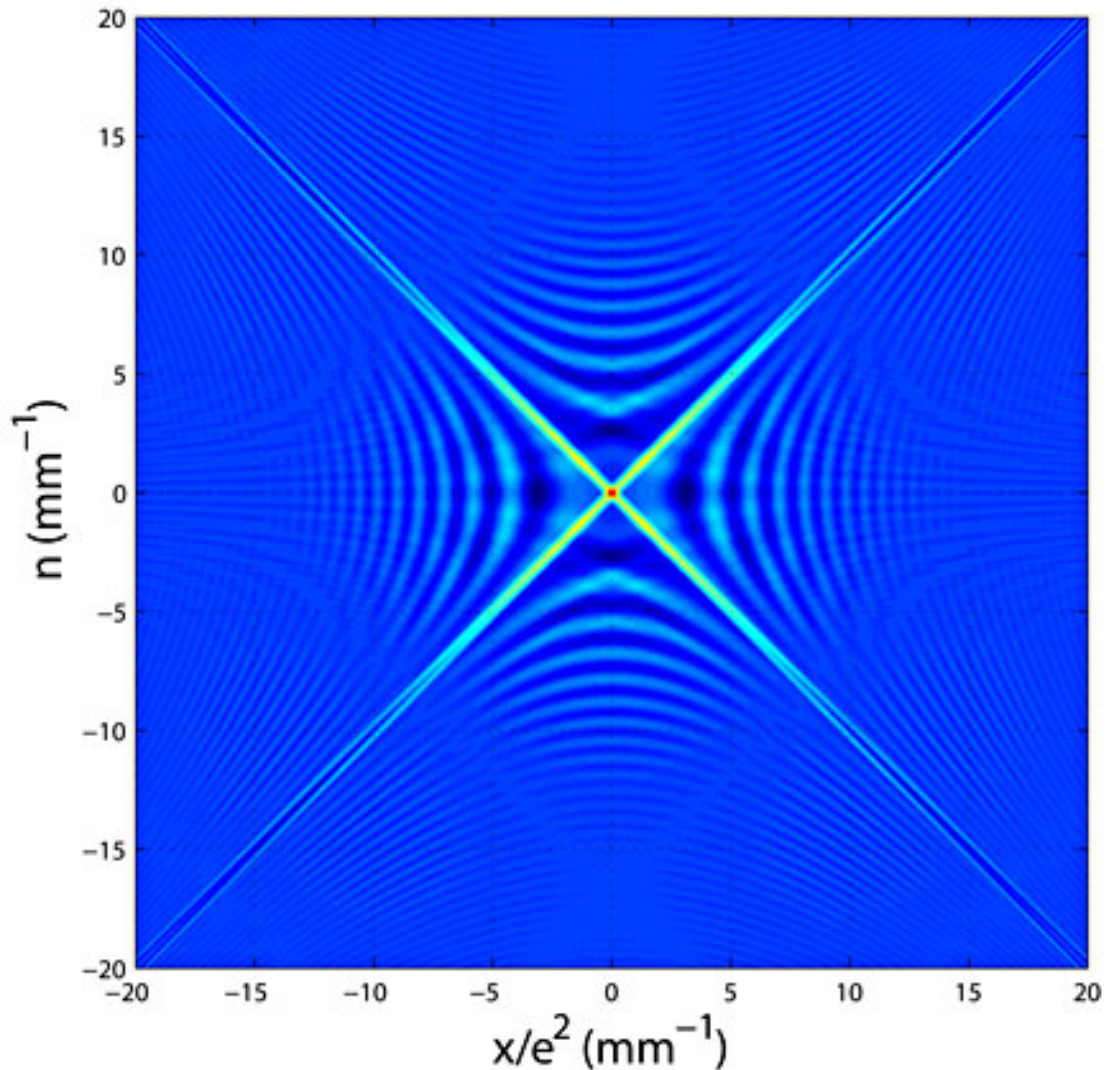


Figure 11

\* Figure 11: Representation of the Wigner distribution of  $I(x)$ , noted  $\mathcal{W}_{I_e}$  for  $z = 100 \text{ mm}$  and  $d = 60 \mu\text{m}$ .

We can note the effects of the autocorrelation term of the function  $\text{rect}(x'/d)$  on the interference terms.

Given the relation linking Wigner distribution functions with fractional Fourier transforms, we shall use this transform to reconstitute fiber holograms.

### 3.2. Fiber holograms reconstituted by fractional FTs

Our aim is to indicate that fractional Fourier transforms allow us to reconstitute hologram images when the best fractional index is chosen. Once again we apply the same argument on the term noted  $S(x)$  and calculate its fractional Fourier transformation:

$$F_\alpha[S(x)](x_a) = C(\alpha) \exp\left(i \frac{\pi x_a^2}{\epsilon^2 \tan \alpha}\right) \int_{-\infty}^{+\infty} \cos\left[\frac{\pi x^2}{\lambda z} - \frac{\pi}{4}\right] F(x) \exp\left(i \frac{\pi x^2}{\epsilon^2 \tan \alpha}\right) \exp\left(-i 2\pi \frac{x_a x}{\epsilon^2 \sin \alpha}\right) dx, \quad (3.17)$$

If equation (3.17) is written as the sum of fractional Fourier transforms, we obtain:

$$F_{\alpha}[S(x)](x_a) = S_{a+}(x_a) + S_{a-}(x_a), \quad (3.18)$$

The calculation of  $S_{a+}$  and  $S_{a-}$  gives the following results:

$$S_{a+}(x_a) = \frac{\exp(-i\pi/4)}{\sqrt{\lambda z}} \exp\left(-i\frac{\pi x_a^2}{\lambda z}\right) F_{\theta+}[F(x)](rx_a), \quad (3.19)$$

and:

$$S_{a-}(x_a) = \frac{\exp(+i\pi/4)}{\sqrt{\lambda z}} \exp\left(i\frac{\pi x_a^2}{\lambda z}\right) F_{\theta-}[F(x)](rx_a), \quad (3.20)$$

where:

$$\tan \theta_{\pm} = \frac{\tan \alpha}{1 \pm \frac{\epsilon^2}{\lambda z} \tan \alpha} \quad \text{et} \quad r = \frac{\sin \theta}{\sin \alpha}. \quad (3.21)$$

Intensities  $S_{a+}$  and  $S_{a-}$  are therefore written:

$$|S_{a+}(x_a)|^2 = \frac{1}{\lambda z} |F_{\theta+}[F(x)](rx_a)|^2, \quad (3.22)$$

and:

$$|S_{a-}(x_a)|^2 = \frac{1}{\lambda z} |F_{\theta-}[F(x)](rx_a)|^2. \quad (3.23)$$

These expressions carry out the Radon projection of  $W_{sa+}$  and  $W_{sa-}$  following an axis  $x_a$  having an angle  $\frac{\pi}{2}$  with the  $x$  axis.

Rewriting  $F_{\alpha}[S(x)](x_a)$  as a function of two FRFT of order  $\theta_+$  and  $\theta_-$  helps its interpretation. In fact, if:

$$\alpha = \alpha_{opt} = \arctan(\lambda z / \epsilon^2) \geq 0, \quad (3.24)$$

Equation (3.21) shows that  $\theta_-$  tends to  $\pi/2$  and therefore  $F_{\theta-}$  becomes a standard Fourier transform.

If we leave out the constant  $1/\lambda z$ ,  $|S_{a_{opt}}(x_{a_{opt}})|^2$  is nothing but the object function  $O(x_a)$ .

It is worth noting that relation (3.24) represents a digital re-focalization on the fiber image considering relation (2.29).

Actually, it corresponds to the cancellation of the phase term owing to the propagation on a distance  $z$ . Thus, the envelope is extracted, but this envelope is used to carry out standard  $FT$ s.

### Attention

The phase term cancelled by an optimal order fractional  $FT$  corresponds to an optimal rotation in the Wigner representation where one of the two branches becomes parallel to the spatial frequency axis  $\nu_u$ . This relation is well-known to turn a frequency drift into a Dirac pulse.

#### a) Simulations

In practice, the intensity level of a diffraction figure background does not have any interest. From a physical viewpoint, it corresponds to a unitary reference wave. The constant term of  $I(x) = A(x)\hat{A}(x)$  will be therefore removed to obtain a mean close to zero.

$I_c$  indicates the corrected function.  $I_{ca}(x_a)$  is its representation in the fractional domain  $\alpha$ . For example, we will use again the curve in figure 9, obtained for  $d = 60 \mu m$ ,  $z = 100 mm$  and  $\lambda = 632.8 nm$  (wavelength). Its Wigner distribution function is represented in figure 11. Let us apply fractional Fourier transform on  $I(x)$ .

We obtain  $|I_{ca}(x_a)|^2$  for which figure 12 illustrates the fiber restitution.

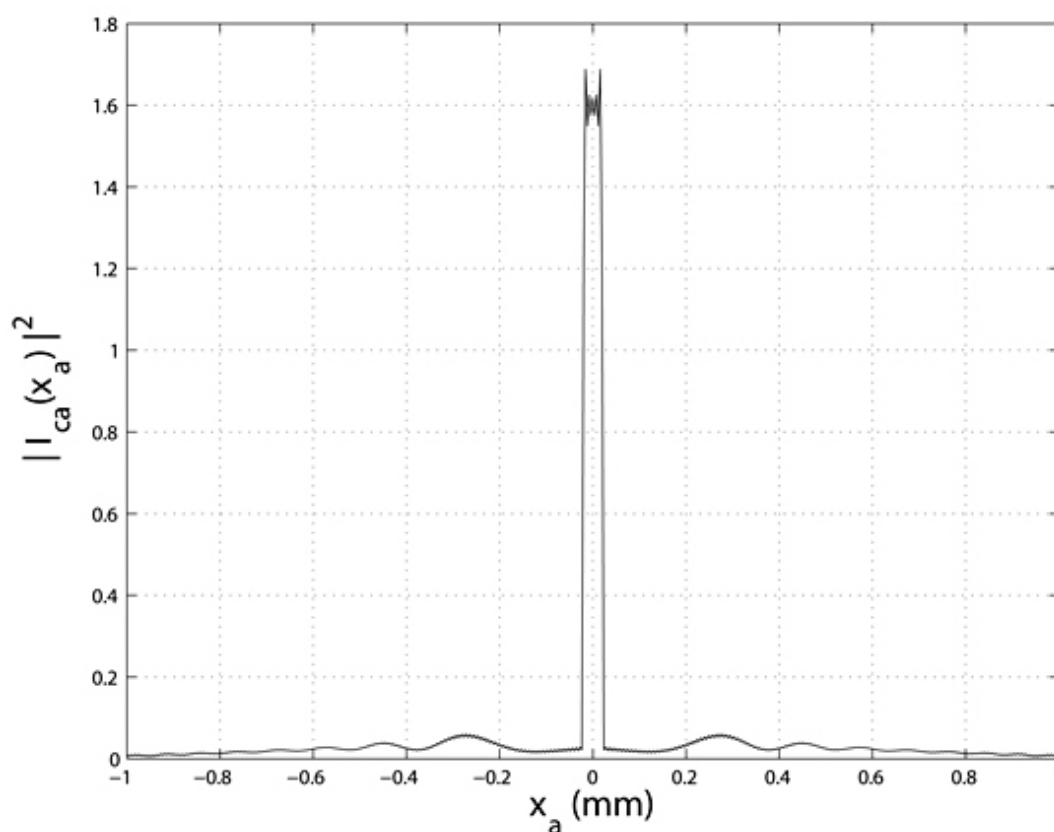


Figure 12

\* Figure 12:  $I_c(x)$  restitution by FRFT for the optimal fractional index 0.5.

This curve is very close to the rectangle function.

The oscillations are linked to the crossed terms of relation (3.13) and the projection of term  $S_{u+}$  on the  $x_u$  axis in Wigner space.

The Wigner distribution of this transform (figure 13) highlights a  $\pi/4$  rotation of  $W_{I_e}$ . This rotation is the best one found thanks to relation (3.24). With this operation, we made the distribution linked to  $S_{a-}$  parallel to the spatial frequency axis. By calculating the square of its module, energy density is optimized by projection. If we had carried out an opposite order  $a = -0.5$ , we would have made the Wigner distribution linked to  $S_{a-}$  parallel to the spatial frequency axis as indicated in figure 13. In this case, we would have focused on real images.

\* \*

\*

We showed that fractional Fourier transforms were well adapted to analyze diffraction figures. The intensity distribution of in-line holograms is partly described by two joint linear chirped functions.

We established a relation allowing us to link the fractional index to the position of the object during the recording.

Hence, we note that, if the frequency drift is cancelled, we focus on the object at the same time.

Finally, thanks to the definition of  $\epsilon^2$ , we synthesized the mathematical definitions of  $FT$ , the optical definition of FRFT and the digitization of optical signals by a digital acquisition system.

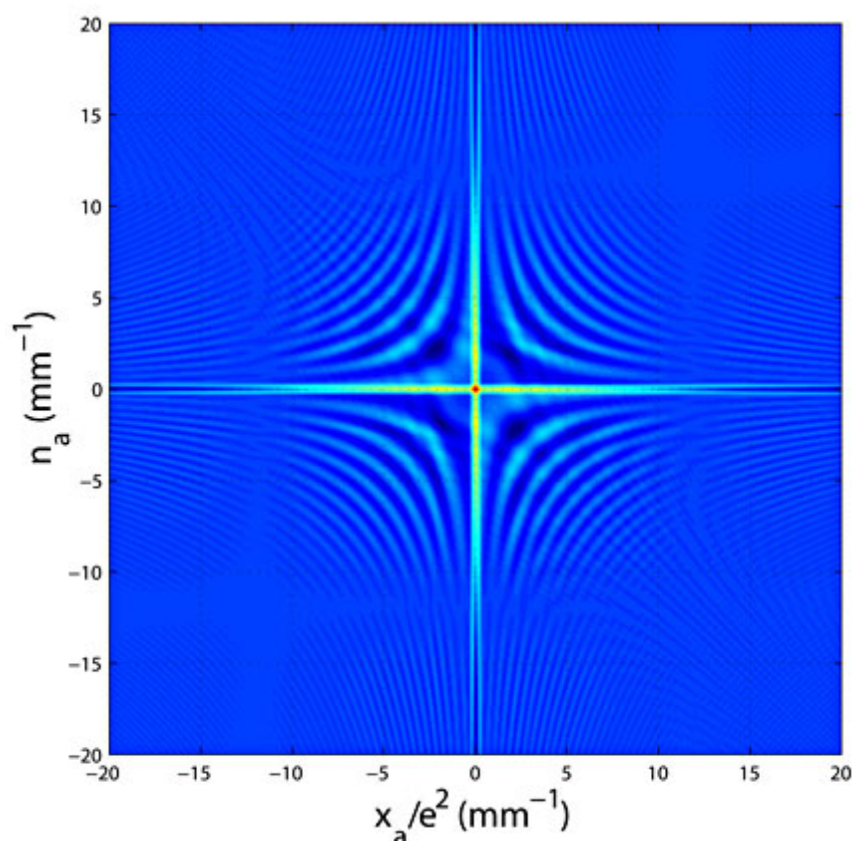


Figure 13

\* Figure 13: Representation of the  $\alpha = \pi/4$  order FRFT Wigner distribution of  $I_{ca}(x_a)$ .

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